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*Some observations for mathematics teachers**

CARROLL V. NEWSOM, *President, New York University, New York, New York.*

"We now have the opportunity, sought for so long, to exercise fundamental leadership in the creation of a better educational program in this country." Some challenges and cautions relative to this leadership are presented here.

A PERSON SENSES the passage of time, even more than through scrutiny of the calendar, by noting changes in his environment. Thus Rip Van Winkle knew that he had slept a very long time, for his surroundings had changed radically during his memorable period of rest. Today I feel very old, for I have appeared before you before, but under vastly different circumstances. In fact, the changes that have taken place in the world since I decided to be a teacher of mathematics are amazing, nay, unbelievable; and the effects of these changes upon our profession, especially in recent years, are equally fantastic.

I had great trouble obtaining a position when I first appeared upon the world scene with my credentials in mathematics. Many times, when applying for a job, I was told by school administrators that it was unfortunate indeed that I was unprepared to teach some useful subject; if I could only coach football. In the end I did coach girls' basketball; I taught chemistry, general science, and public speaking, coached all the school plays, and did substitute teaching in home economics, thereby teaching seven classes a day, to make up a package deal that permitted me to teach a few classes in mathematics. My first full-time position as a university instructor involved my teaching mechanical drawing, physics, astronomy, two classes in college algebra, trigonometry, and advanced calculus—a total of twenty-one hours a week. Moreover, it was made clear when I accepted the position that any future promotion would depend essentially on the amount of research that I produced.

Gone are the days, I hope forever, when we must tolerate such statements as, "Mathematics, along with the classical languages, is a dead subject"; "Sound training in good citizenship is the basic purpose of the high school"; "Mathematics is good only for its disciplinary value"; "Some mathematics should be taught as a background for those who may study engineering"; "No mathematics should be required; let it be chosen by the very few who need it"; "Probably mathematics is the only subject in which nothing more remains to be done"—a statement made by a distinguished anthropologist in the 1920's. I still recall quite distinctly a statement made in the first days of World War II by one of our most renowned American mathematicians in which he asserted his belief that mathematics really had nothing to contribute to the war effort; I was one of the very few who had the temerity to take issue with him.

* Presented at the eighteenth Christmas meeting of The National Council of Teachers of Mathematics, in New York City, December 30, 1958.

Yes, mathematics is now at the very peak of its popularity, probably for all of its history. This fact provides us great satisfaction, but it also involves the acceptance of grave responsibility. Mathematicians must now, as never before, be statesmen as they provide guidance to an uncertain public. Extremists, always ready to take advantage of every opportunity to advocate their ideas, must be identified as such. The doctrines of professional innovators, always so eager for publicity, must be regarded with due caution. We now have the opportunity, sought for so long, to exercise fundamental leadership in the creation of a better educational program in this country; it would be tragic indeed if we should not be equal to the task.

It is essential, first, that mathematics receive proper definition in the minds of the educated. Let it be taught that mathematics reveals human wisdom at its best, that the symbolic models of mathematics —truly human creations—must be classified along with the finest of literature, art, and philosophy when describing the humanities; in fact, mathematics is one of the humanities, not one of the sciences. It is true, of course, that modern science, especially physical science, has had great success in studying relationships within nature by using mathematical models as co-ordinating mechanisms; this fact enhances the prestige of mathematics, and that of its creators, but it has no great relevance when considering the actual nature of mathematics. One may predict with considerable certainty that mathematical models will soon be used extensively in the social sciences; in truth, we know of no alternative to the doctrine that their utilization must be fundamental in all of man's efforts to comprehend and control his environment. Moreover, and this is most important, we shall find that studies of possible inherent limitations upon the human mind as an interpreter of the structural properties of nature will be concerned with many aspects of the field of knowledge presently considered under the title, "Foundations of Mathematics"; that is, one who seeks to understand the actual meaning and nature of human knowledge, other than that which is purely descriptive, will be obligated to study mathematics, its concepts, and its methods.

If one accepts my beliefs on such matters, it is apparent that certain common practices need modification. First, although I believe sincerely that most mathematicians can profit from varied experiences that involve mathematical application, it does not follow that a knowledge of some kind of science is a substitute for a knowledge of mathematics, or vice versa. When a school or college specifies electives by groups, as part of the prescribed graduation requirement, mathematics can be listed properly only among the humanities. Second, the study of mathematics is one of the two or three basic elements in any school or college program designed to provide a general education. Those who disagree, I find, were reared on a diet of useless algebraic and numerical manipulation, so they are unfamiliar with the actual nature of mathematics, its uses, and its values. Let us not duplicate such an unfortunate educational experience for future generations of citizens, including perhaps some school administrators, who in turn may be called upon to pass judgment upon the significance of mathematics. In brief, let's teach mathematics and not merely a strange assortment of its devices, most of which are of little value in real mathematical experiences.

Every student should have an opportunity to work with simple mathematical systems, or models. It would be my personal judgment that first priority might well be given to the study of the concept of integer as characterized by a set of primitive propositions and to the study of the concept of rational number as similarly characterized. These concepts are not too difficult, and acquaintance with them provides excellent background for additional

studies, especially those that are necessary in traditional algebra and analytic geometry, and for the many experiences of life that involve magnitudes. Moreover, it seems to me, such studies are of greater immediate significance to students than, for example, the development of an algebra of sets, even admitting the fundamental nature of the concept of set. For able students of broad scholarly interests, of course, there should be additional early experience with such mathematical systems as those which characterize Boolean algebra, the group, the real number system, the complex number system, the Euclidean plane, non-Euclidean geometry, and so on. I suspect that algebra, trigonometry, and geometry, as such, should disappear as separate and distinct segments of our mathematical studies.

Parenthetically, I may remark that nowhere in mathematics has the strength of tradition been more apparent than in our teaching of trigonometry. The sine and cosine actually make their appearance in mathematics as solutions of a second-order differential equation; their use and understanding require the same order of maturity as, for example, the Bessel Functions and the Legendre Functions. For many decades, however, we have taught a very insignificant application of the trigonometric functions, that is, to triangles, and then have indulged in some hocus-pocus in the hope that students would use the functions in other circumstances having nothing to do with angles or triangles and not ask embarrassing questions. Naturally there has been utter confusion. Although some brief study of trigonometric ratios can be a part of the study of similar triangles in the Euclidean plane, a systematic study of the trigonometric functions must be postponed until students are prepared to deal with the appropriate differential equation. It is quite apparent to me that the study of some kinds of elementary differential equations should precede much of what now appears in a first course in calculus. A tremendous challenge to curriculum designers is inherent in all these considerations.

As I advocate a better brand of mathematics teaching, I feel obligated to issue a word of warning, especially in the light of certain ideas that are being advocated—usually by accomplished mathematical scholars. In fact, may I quote at this point from some remarks that I made before this organization some twelve years ago:

> In presenting the material of mathematics the good teacher will always remember that rigor is a relative term. As Felix Klein so often pointed out, that which is best from a pedagogical point of view is not always the most logical by the teacher's standards. Even though mathematics, by definition, is a deductive science, its presentation to students and its natural development will be more inductive than deductive. Mathematicians are too sensitive on this score; an author of some important research will publish an elegant deductive demonstration of his results, but he feels it is unethical to reveal the long and laborious process, usually inductive, that has preceded the ultimate deductive organization. For too long we have made the mistake of plunging the student of plane geometry into a discipline highly organized along deductive lines. Of course, the deductive should be shown and studied but that type of organization reveals the subject matter of the course in its most advanced stage of development. By contrast, most elementary courses in algebra sin in the reverse direction by never showing any part of the ultimate logical organization which mathematicians find to be necessary.
>
> Every teacher must keep in mind that mathematics in a technical sense is a triumph of applied logic, but the devices of logic are not simple, they are not inherited, and the teacher of mathematics must take each student by the hand as he leads him through the complexities that are involved. As teachers, let us not forget that genuine mathematical understanding cannot precede the student's comprehension of the logical and his ability to read and understand.
>
> A good teacher cannot afford to ignore the historical background of his subject. The biologists have their theory of recapitulation which states that to a great extent each individual in his early development reproduces in himself the early evolution of the race. I sincerely believe, more than we sometimes realize, that each student develops in his understanding of mathematics along much the same line that history reveals that man developed. It is apparent, of course, that too rigid an interpretation can be placed upon this statement, but I, as a teacher, must admit my own indebtedness to my study of man's historical struggle to build the remarkable system called mathema[illegible]. A knowledge of

history is imperative for other reasons. The evolution of such a discipline as mathematics cannot be separated from the rest of the stream of history; mathematics has been a contributor and a recipient in terms of the other factors of civilization. The fact that mathematics is a living body should dominate the attitudes of the teacher and should be portrayed to our students.

May I conclude my present remarks with still another word of caution. In our day of triumph, when the advice of mathematicians is respected and is being sought, let us remember constantly that we are only part of the world of scholarship and that we are citizens of the world. Our destiny is that of all scholarship and of all citizens. As Paul Sears has said, "The applications of science must be guided, managed, controlled according to ethical and aesthetic principles and in light of our most profound understanding. . . . As the familiar lines have it, for better or for worse, for richer or poorer, in sickness and in health, our democracy is committed to the principle that each of us has the privilege of helping to shape policy. Policy implies planning. Planning, a public obligation that rests upon the meeting of minds, will determine whether the problems of terrestrial space in which we live will receive the priority they deserve, whether the United States can continue to maintain its freedoms, its present world leadership, and its power to help others."

The challenge of the day to all of us and to this organization is very great. Let us face it boldly and courageously.

"The most significant thing in education today is the wide recognition of individual differences in ability among pupils. This is particularly true in regard to the high school and is due partly to a realization that our secondary school population is very different from that of thirty years ago . . ."—*W. D. Reeves in* THE MATHEMATICS TEACHER, *December, 1924.*

Algebra is a branch of mathematics that has been transformed by the mathematical research of the last quarter-century. A first-year graduate school course in algebra today bears little or no resemblance to that of 30 years ago. The key concepts of the earlier course would now be subsumed under newer and broader concepts that had not been formulated in 1928. Yet the corresponding course of 30 years ago differed little or not at all from its predecessor of 1900."—"Program for college preparatory mathematics," *Report of the Commission on Mathematics, College Entrance Examination Board, New York, 1959, p. 2.*

A simple example of a noncommutative algebra

ARNOLD WENDT, *Western Illinois University, Macomb, Illinois.*

Students may not consider the commutative principle very important if they never see an example where multiplication is noncommutative.

Introduction

NOWADAYS TEACHERS are encouraged to give more attention to the postulational nature of algebra. Often this emphasis takes the form of an abstraction of a notion with which the students are already familiar. Many students remain singularly unimpressed by attempts to postulate or prove an idea they have long ago accepted as a fact holding universally in mathematics. So long as we remain in the algebra of the real or complex numbers, the same old rules apply.

We have found that one way to get students to appreciate more fully our rules of arithmetic is to expose them to an easily understandable, yet mathematically important, example of an algebra in which not all the usual laws hold. Such an example is the algebra of 2×2 (read 2 by 2) matrices with real numbers as elements. They are easily understandable, because in performing operations with or on these matrices we make use of ordinary arithmetic. Much of what follows can be readily followed by high school students, if not all of it.

For future reference let us first list the usual laws of arithmetic satisfied by the real numbers. For all real numbers a, b, c:

1. *Closure laws:* $a+b$ and $a\times b$ are unique real numbers.
2. *Commutative laws:* $a+b=b+a$ and $a\times b=b\times a$.
3. *Associative laws:* $a+(b+c)=(a+b)+c$ and $a\times(b\times c)=(a\times b)\times c$.
4. *Distributive law:* $a(b+c)=(a\times b)+(a\times c)$.
5. *Identity elements:* There exists a number 0 such that $a+0=a$ for all a, and there exists a number 1 such that $a\times 1=a$ for all a.
6. *Inverse elements:* For each number a there exists a number $(-a)$ such that $a+(-a)=0$, and for each $a\neq 0$ there exists a number a^{-1} such that $a\times a^{-1}=1$.

From these first six laws we can prove the following law, written in two equivalent forms.

7. *Cancellation law:* If $a\times b=a\times c$ and $a\neq 0$, then $b=c$.

7′. (Alternate form): If $a\times b=0$, then $a=0$, or $b=0$, or both.

The laws one through six are usually referred to as postulates for a *field*. They are satisfied by other mathematical entities besides the real and complex numbers.

Matrices

We now define a new kind of mathematical entity called a *matrix*, plural *matrices*.

Definition: A 2×2 matrix is a square array of real numbers represented by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The subscripts merely denote the row and column, in that order, in which the number is located. For example, a_{21} means the number or *element* in the second row, first column.

Remarks: 1. Although similar in appearance to a determinant, a matrix is not the same thing as a determinant. *A determinant* (we are assuming the reader is familiar with determinants) *is a number* obtained from a square array of numbers by combining them according to certain rules. On the other hand, a matrix is the array of numbers itself. There is no rule for combining the elements in a matrix.

2. Since the laws in which we are going to be interested can be demonstrated most easily by using 2×2 matrices, we shall confine our attention chiefly to them. However, a matrix may have any number of rows and columns and need not be square, i.e., number of rows need not equal number of columns.
3. The elements in a matrix may be almost any mathematical entity, including even matrices themselves.
4. There is no geometrical representation for general matrices as there is for real numbers and complex numbers. One can consider matrices as a generalization of the notions of complex number and vector. They are sometimes referred to as hypercomplex numbers.
5. The fact that we cannot draw a picture of a matrix does not mean matrices have no application in practical problems. The development of the theory of matrices preceded their application to practical problems, as has been the history of many other mathematical systems. Their theory had its beginnings in the 1850's and so can be considered modern mathematics.

Definition: Two matrices will be considered equal if, and only if, the elements in corresponding positions are equal.

Definition of Addition:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}.$$

Example: Let

$$A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}.$$

Then

$$A+B = \begin{pmatrix} -1+4 & 3+(-1) \\ 1+0 & 2+2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

Since addition of real numbers is commutative and associative, it is not difficult to see that laws 1, 2, 3 of the real numbers under $+$ are satisfied by matrices under $+$.

Also, we have the identity element under addition of Law 5, for the matrix

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfies this property.

Definition of Multiplication:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}.$$

Before checking the properties of matrices under multiplication, we shall amplify the definition above. By noting the pattern in the subscripts, we see that the element in the first row, second column of the product matrix, is obtained by adding the products of corresponding elements in the first row of the left factor and the second column from the right factor. Corresponding elements are determined by proceeding from left to right in the row and from top to bottom in the column. Schematically,

$$\begin{pmatrix} \longrightarrow \\ * \quad * \end{pmatrix} \times \begin{pmatrix} * \\ * \\ \downarrow \end{pmatrix} = \begin{pmatrix} * & N \\ * & * \end{pmatrix}.$$

Using numbers,

$$\begin{pmatrix} \longrightarrow \\ 1 \quad 3 \\ * \quad * \end{pmatrix} \times \begin{pmatrix} * & 1 \\ * & -1 \\ & \downarrow \end{pmatrix} = \begin{pmatrix} * & -2 \\ * & * \end{pmatrix}.$$

This "over and down" rule is a convenient mnemonic device for matrix multiplication. In general, to get the element in the ith row and jth column of the product matrix we proceed from left to right in the ith row of the *left* factor and down the jth column of the right factor, forming products of corresponding elements determined in this manner. Then the products are added to get the entry.

Example:

$$\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(4)+(3)(0) & (1)(1)+(3)(-1) \\ (-2)(4)+(2)(0) & (-2)(1)+(2)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 \\ -8 & -4 \end{pmatrix}.$$

It follows from the definition of multiplication that matrices cannot be multiplied unless the number of columns in the left factor is the same as the number of rows in the right factor.

That matrices are closed under multiplication should be evident. We check the commutative law by performing two more multiplications in detail.

$$\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)(4)+(3)(0) & (-1)(-1)+(3)(2) \\ (1)(4)+(2)(0) & (1)(-1)+(2)(2) \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 7 \\ 4 & 3 \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (4)(-1)+(-1)(1) & (4)(3)+(-1)(2) \\ (0)(-1)+(2)(1) & (0)(3)+(2)(2) \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 10 \\ 2 & 4 \end{pmatrix}.$$

And we see that matrix multiplication is *not* in general commutative. But this does not mean matrix multiplication is useless.

The associative law for matrix multiplication holds, but we shall not prove this fact. We merely illustrate the rule.

$$\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \times \left[\begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \right]$$

$$= \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 17 \\ 10 & 13 \end{pmatrix}.$$

$$\left[\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} \right] \times \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 7 \\ 4 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 17 \\ 10 & 13 \end{pmatrix}.$$

The distributive law also holds; but since multiplication is not commutative, we have a right distributive law and a left distributive law. That is, for matrices we do *not* have in general

$$A(B+C) = (B+C)A.$$

The identity element for multiplication exists and is denoted by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The reader may verify that I has the property of a multiplicative identity. In doing so he will note that I commutes with every matrix and so is both a right and left identity.

We have seen that every matrix has an additive inverse and now investigate the existence of multiplicative inverses. Suppose

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

is a matrix for which we wish to find an inverse. Then it must be true that

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Applying the definition of matrix multiplication we see it is necessary that

$$\begin{pmatrix} 3b_{11}-b_{21} & 3b_{12}-b_{22} \\ b_{11}+2b_{21} & b_{12}+2b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since matrices are equal only if corresponding elements are equal, we have

$$3b_{11} - b_{21} = 1 \quad 3b_{12} - b_{22} = 0$$
$$b_{11} + 2b_{21} = 0 \quad b_{12} + 2b_{22} = 1.$$

These two pairs of simultaneous equations have the solutions $b_{11} = 2/7$, $b_{21} = -1/7$, $b_{12} = 1/7$, $b_{22} = 3/7$, so the proposed inverse is

$$\begin{pmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{pmatrix}.$$

Checking, we see that

$$\begin{pmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{pmatrix} \times \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{pmatrix},$$

so that inverses are both right and left inverses.

While it is true that every real number except zero has a multiplicative inverse, it is not true that every matrix other than the zero matrix has an inverse. For example, let us try to find the inverse of

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

by the method used previously. Then

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

results in the equations

$$a_{11} + 2a_{21} = 1 \quad a_{12} + 2a_{22} = 0$$
$$2a_{11} + 4a_{21} = 0 \quad 2a_{12} + 4a_{22} = 1.$$

We see both sets are inconsistent, and so it is simply not possible to find an inverse for this matrix.

We have one more law to check, the cancellation law for multiplication. That it does *not* hold follows immediately from the following example:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 6 & 2 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Consequently, we cannot say that simply because the product of two matrices is the zero matrix, at least one of the two factors must be the zero matrix. We extend the above example to present a counter example to the first form of Law 7, also. Since

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 6 & 2 \\ -3 & -1 \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix},$$

but

$$\begin{pmatrix} 6 & 2 \\ -3 & -1 \end{pmatrix} \neq \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}.$$

Going very deeply into matrix theory is not the primary purpose of this paper. However, the observant reader's curiosity may be aroused by the fact that previously we saw the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

had no multiplicative inverse, and now we see it is a proper divisor of zero. Is there any connection between these two observations? The answer is yes. Suppose the matrix A is a proper divisor of zero, i.e., $A \times B = 0$ for some matrix B where B is *not* the zero matrix, and suppose A has an inverse A^{-1}. Then

$$A^{-1} \times (A \times B) = (A^{-1} \times A) \times B = I \times B = B,$$

and also

$$A^{-1} \times (A \times B) = A^{-1} \times 0 = 0.$$

Hence B would be equal to the zero matrix, contrary to the assumption that B is not the zero matrix. Consequently, any proper divisor of zero does not have an inverse. The converse is also true.

Summarizing, we see matrix algebra is the same as the algebra of the real numbers with the following exceptions:

1. Multiplication is not commutative.
2. Not all non-zero matrices have an inverse.
3. The cancellation law for multiplication does not hold.

The algebra of matrices with real elements is an example of a "ring with unit element."

Applications

Despite what at first might appear to be shortcomings, matrices have proved handy things to have around. They are used in every branch of pure mathematics. To attempt to list all their applications in practical problems would be impossible. They are used in physics, chemistry, many branches of engineering, psychology, biology, sociology, economics, game theory, linear programming, and statistics. Matrices are often used because they afford a compact form for recording data.

Suppose we look at a trivial example of an application to a problem in higher finance.

Number of Units Consumed

	Soda	Ice Cream	Candy
John	2	1	3
Mary	1	2	0
Jim	1	1	1

Cost per Unit

Soda	10 cents
Ice Cream	8 cents
Candy	5 cents

We form the matrix product

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 10 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} 43 \\ 26 \\ 23 \end{pmatrix}$$

and leave to the reader the interpretation of the product matrix. The product

$$(1, 1, 1) \times \left[\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 10 \\ 8 \\ 5 \end{pmatrix} \right]$$

$$= (1, 1, 1) \times \begin{pmatrix} 43 \\ 26 \\ 23 \end{pmatrix} = (92)$$

also has an interpretation which the reader should check.

Not all important applications of matrices involve matrix algebra. Often their advantage lies in the compact manner in which the matrix notation can represent information. We note, for example, that the equations

$$\begin{aligned} x+2y-3z&=6 \\ 2x+5y-2z&=4 \\ x+3y-\ z&=0 \end{aligned}$$

can be represented by the matrix

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 2 & 5 & -2 & 4 \\ 1 & 3 & -1 & 0 \end{pmatrix}.$$

By performing operations, which correspond to eliminating unknowns, on the rows of this matrix one can obtain an equivalent set of equations represented by

$$\begin{pmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The numbers in the last column are the solutions of the equations.

Before leaving this example we should point out that if one had a set of equations with the same coefficients as above, i.e., equations of form

$$\begin{aligned} x+2y-3z&=C_1 \\ 2x+5y-2z&=C_2 \\ x+3y-\ z&=C_3, \end{aligned}$$

then application of matrix algebra would give a convenient method for finding solutions once the inverse of the coefficient

matrix has been determined. In the present case the inverse of

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & -1 \end{pmatrix} \text{ is } \begin{pmatrix} -1/2 & 7/2 & -11/2 \\ 0 & -1 & 2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix},$$

that is, their product in either order is I. Since the above equations may be represented by

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & -1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix},$$

multiplying both sides on the left by the inverse gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 & 7/2 & -11/2 \\ 0 & -1 & 2 \\ -1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

Thus, substituting different sets of values for the C's will automatically grind out the answers upon performing the indicated multiplication.

Isomorphism

We conclude with a few remarks having no bearing on the body of this paper, but which, in the writer's opinion, may amplify the concept of a matrix as a number.

In the sense that certain subsets of matrices are nothing more than the real numbers and complex numbers in disguise, matrices are an extension, or generalization, of these two number systems. The reader is probably familiar with the following hierarchy:

Complex numbers $\supset$ real numbers $\supset$ rational numbers $\supset$ integers and 0. The symbol $\supset$ means "contain." What we have said above is merely that we can also write

Matrices$\supset$complex numbers$\supset$, etc.

Considering only 2×2 matrices is sufficient for our purposes.

We have already noted that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

behaves like the additive identity 0 for the reals and that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

behaves like the multiplicative identity 1 for the reals.

Looking at these two correspondences

$$\begin{matrix} 0 \\ \updownarrow \end{matrix} \quad \begin{matrix} 1 \\ \updownarrow \end{matrix}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

suggests that for any real number r we set up the correspondence

$$\begin{matrix} r \\ \updownarrow \end{matrix}$$
$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}.$$

Under this correspondence sums and products of corresponding elements again correspond to each other.

Example:

$$\begin{matrix} 4 \\ \updownarrow \end{matrix} \quad + \quad \begin{matrix} 3/4 \\ \updownarrow \end{matrix} \quad = \quad \begin{matrix} 4\,3/4 \\ \updownarrow \end{matrix}$$
$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix} = \begin{pmatrix} 4\,3/4 & 0 \\ 0 & 4\,3/4 \end{pmatrix}.$$

$$\begin{matrix} 4 \\ \updownarrow \end{matrix} \quad \times \quad \begin{matrix} 3/4 \\ \updownarrow \end{matrix} \quad = \quad \begin{matrix} 3 \\ \updownarrow \end{matrix}$$
$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \times \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

Now the complex numbers were formed from the reals by forming pairs of reals and adding $\sqrt{-1}$ to the system in a special way, namely, $a+ib$, where a and b are real numbers. To extend our correspondence of matrices to the complex numbers we will need a matrix to represent i. It turns out that the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow i$$

will complete our correspondence. For we see

$$\begin{matrix} i^2 \\ \updownarrow \end{matrix} \quad = \quad \begin{matrix} -1 \\ \updownarrow \end{matrix}$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\begin{array}{ccccccc} a & + & i & \times & b & = & a+ib \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} & + & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \times & \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} & = & \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \end{array}$$

The correspondence

$$a+ib \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

shows that the complex numbers may be represented by 2×2 matrices with real elements. The reader may check that sums and products are preserved under this correspondence. A one-to-one correspondence preserving sums and products is called an "isomorphism."

Other noncommutative systems

There are other important, noncommutative systems in mathematics. Noncommutative groups are such a system. For an example the reader is referred to a recent article* in The Mathematics Teacher.

* Carl H. Denbow, "To Teach Modern Algebra," The Mathematics Teacher, XII (March 1959), 162–170.

Another example is the "cross product" of two vectors. Here we have

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

There is also a "dot product," denoted by $\vec{A} \cdot \vec{B}$, for vectors that is commutative, however. Vectors are especially important in mechanics.

The quaternions are another example of a noncommutative system, though they are not as important as the others mentioned. They are numbers formed from pairs of complex numbers in much the same way complex numbers are formed from pairs of real numbers. For further information on any of the subjects mentioned in this paper, the reader is referred to the readable books listed below. Information on vector products may be found in a great variety of texts on vectoral mechanics and vector analysis, and in some analytic geometry texts.

Suggested References

Andree, Richard V. *Selections from Modern Abstract Algebra*. New York: Henry Holt and Co., 1958.

Birkhoff, Garrett, and MacLane, Saunders. *Survey of Modern Algebra* (Rev. ed.). New York: The Macmillan Co., 1953.

Hohn, Franz E. *Elementary Matrix Algebra*. New York: The Macmillan Co., 1958.

From THE MATHEMATICS TEACHER *of Thirty Years Ago*

"Certain tendencies which have long been operating in our schools have by their cumulative effect brought about a change in the ninth grade school population which has made classification into ability groups and other devices for adjusting instruction to the varying needs of the individual pupil more important than ever before."—*L. E. Mensenkamp*, "Ability Classification in Ninth Grade Algebra."

Primitive Pythagorean triples

BEN MOSHAN, *New York, New York.*

An ancient problem continues to fascinate and challenge. High school students know about 3, 4, 5 right triangles. Do they know about 8, 15, 17 right triangles and 861, 620, 1061 right triangles?

THE PROOF THAT the area of the square on the hypotenuse of a right triangle is equal to the sum of the areas of the squares on the legs is traditionally ascribed to Pythagoras, who lived more than 2000 years ago. Conversely, if a, b, and c are three line segments such that

$$\text{(I)} \qquad a^2+b^2=c^2,$$

then the three line segments will form a right triangle.

Euclid established a formal proof for finding two square numbers (integers) such that their sum is always a square integer. Referring to Book X, Heath's *Euclid*, Cambridge, At The University Press, lemmas 1 and 2 of Theorem 28, expressing Euclid's conclusion algebraically, these values of a^2, b^2, and c^2 must always be of the form

$$a^2=mp^2\cdot mq^2, \quad b^2=\left(\frac{mp^2-mq^2}{2}\right)^2,$$

$$c^2=\left(\frac{mp^2+mq^2}{2}\right)^2,$$

with the further condition that mp^2 and mq^2 must both be either even or odd simultaneously.

This relationship, $a^2+b^2=c^2$, when a, b, and c are integers, may be called a Pythagorean triple, and furthermore, if the greatest common divisor (g.c.d.) of a, b, and c is unity, then the triple is said to be primitive. For example, consider the Pythagorean triple $6^2+8^2=10^2$. Since the g.c.d. is >1, this triple is not primitive, whereas the triple $3^2+4^2=5^2$ is primitive. The problem to be considered here is that of developing methods for finding all such primitive triples.

Referring to a more modern exposition on this subject, Rademacher and Toeplitz in their book *The Enjoyment of Mathematics*, Princeton University Press, pages 88–93, prove that the following values of a, b, and c always satisfy equation (I):

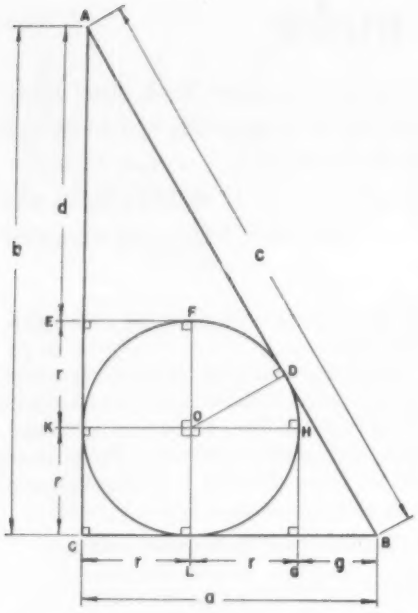
(A) $a=u^2-v^2$ (a is always odd),
(B) $b=2uv$ (b is always even),
(C) $c=u^2+v^2$ (c is always odd).

They also prove that u and v must always satisfy the following conditions (which they call "reduced solutions") in order that the resulting triples will always be primitive:

(D) u and v are "relatively prime" integers, i.e., they have no common divisor >1,
(E) u and v are of "opposite parity," i.e., if u or v is even then the other must be odd,
(F) $u>v$ to ensure that a is positive.

The purpose of this manuscript is to present a new and different technique for finding all such primitive Pythagorean triples. This method makes use of the radius of the circle inscribed in the right triangle, and will be called the "inradius" method.

While working and developing the inradius method, it became apparent that all Pythagorean triples, whether primitive or not, could be grouped and classified. Equations relating to this method of classification, and a table of these groupings, restricted to primitive triples, are included in the last section.



Triangle ABC is a right triangle and r is the radius of the inscribed circle, known as the inradius. Draw the tangents to the circle, $FE \perp AC$ and $GH \perp BC$.

1. From the diagram the following are evident:

(1) $\qquad c^2=a^2+b^2.$

(2) $\qquad a=2r+g \quad$ or $\quad g=a-2r.$

(3) $\qquad b=2r+d \quad$ or $\quad d=b-2r.$

Now, $c=AD+BD=AK+BL$, since the tangents from an external point to a circle are equal. But $AK=d+r$ and $BL=g+r$. Therefore,

(4) $\qquad c=2r+g+d.$

By substituting (2) and (3) in (4) we have

(5) $\qquad c=a+b-2r.$

From (2) and (5) it follows that

(6) $\qquad g=c-b.$

From (3) and (5) it follows that

(7) $\qquad d=c-a.$

Now, if we substitute (2), (3), and (4) in (1) we have

$$(2r+g+d)^2=(2r+g)^2+(2r+d)^2,$$

from which we arrive at

(8) $\qquad gd=2r^2.$

2. We know that there are integer values of a, b, and c which satisfy the equation (1), and thus, if a, b, and c are integers, it is obvious from (5), (6), and (7) that $2r$, g, and d are integers. And since g and d are integers, it follows from (8), $2r^2$ is also an integer.

It will now be proved that since both $2r$ and $2r^2$ are integers, r is also an integer. For, if r is not an integer, and since $2r$ is, then r must be of the form $r=k+\frac{1}{2}$, where k is an integer. It then follows that

$$2r^2=2(k+\tfrac{1}{2})^2=2k^2+2k+\tfrac{1}{2},$$

which is absurd since $2r^2$ is an integer.

3. It will now be proved that if the triple is primitive, then g and d are relatively prime and of opposite parity. For if g and d have a common divisor $k>1$, then every divisor of k would be a common divisor of g and d, or even if k is prime we can express g and d as

$$g=kg_1, \qquad d=kd_1.$$

Equation (8) would then be

$$k^2g_1d_1=2r^2 \quad \text{or} \quad g_1d_1=2(r/k)^2,$$

thus k is a factor of r, say $r=kr_1$.

Then from (2), (3), and (4) we have

$$a=2kr_1+kg_1 \qquad \text{or } (a/k)=2r_1+g_1$$

$$b=2kr_1+kd_1 \qquad \text{or } (b/k)=2r_1+d_1$$

$$c=2kr_1+kg_1+kd_1 \text{ or } (c/k)=2r_1+g_1+d_1.$$

Thus, k divides a, b, and c, which is contrary to the fact that the triple is primitive.

Further, since $gd=2r^2$, which is an even integer, either g or d is even and the other is odd. For if g and d are both even, then g and d are not relatively prime.

4. Every composite integer r can be expressed as a product of primes in one and only one way if we do not distinguish be-

tween arrangements of the same prime factors, say,

$$r = p_1^u \cdot p_2^v \cdot p_3^w \cdot \cdots \cdot p_n^z,$$

where $p_1, p_2, p_3, \cdots, p_n$ are distinct primes and $u, v, w, \cdots, z$ are integers.

Thus, we can express $2r^2$ as a product of distinct primes,

$$2r^2 = 2(2^t \cdot p_1^u \cdot p_2^v \cdot \cdots \cdot p_n^z)^2 = gd,$$

where $p_1, p_2, \cdots, p_n$ are distinct odd primes only and $t, u, v, \cdots, z$ are integers.

All of the factors of $2r^2$, since $2r^2 = gd$, must be in g and d taken together, but each distinct prime can only be in g or d since g and d are relatively prime. Therefore, $2(2^t)^2$ can only appear in either g or d, and since all of the other distinct primes of $2r^2$ are odd, it follows that either g or d is even and the other is odd. For convenience, let g be odd and d even, so that we can let $g = (G)^2$, where G is the product only of distinct odd primes in $2r^2$, thus, G is always odd. Then, we can let $d = 2(n)^2$, where n is the product of the remaining distinct primes in $2r^2$, whether even or odd, and G and n are relatively prime.

Then from (8) it follows

$$gd = 2r^2 = 2n^2G^2, \tag{9}$$

and

$$r = nG. \tag{10}$$

From (2), (3), and (4) we now have

$$a = 2r + g = 2nG + G^2. \tag{11}$$

$$b = 2r + d = 2nG + 2n^2. \tag{12}$$

$$c = 2r + g + d = 2nG + G^2 + 2n^2. \tag{13}$$

5. We have found the necessary conditions for a primitive triple. We will now prove that the values of a, b, and c, as expressed in equations (11), (12), and (13), are always primitive when G is odd and n and G are relatively prime.

By simply substituting these values in equation (1) we have

$$(2nG + G^2)^2 + (2nG + 2n^2)^2 = (2nG + G^2 + 2n^2)^2$$

which when multiplied out satisfies equation (1).

The next condition for a primitive triple is that G be odd. For if G is even, it is apparent from (11), (12), and (13) that a, b, and c are also all even and the resulting triple is not primitive.

It now remains to prove that the triple is always primitive when n and G are relatively prime. Since G must be odd, from (11), (12), and (13) we see that a and c must be odd and b must be even, and as such they do not have the common factor 2. We will now show that they do not have an odd common factor $k > 2$. For if they did then every prime divisor of k would be a common divisor of a, b, and c, or even if k is prime we can let

$$a = ka_1, \qquad b = kb_1, \qquad c = kc_1$$

and then from (4), (2), and (3) we have

$$kc_1 = 2nG + G^2 + 2n^2 \qquad kc_1 = 2nG + G^2 + 2n^2$$
$$ka_1 = 2nG + G^2 \qquad kb_1 = 2nG + 2n^2$$

Subtract,

$$k(c_1 - a_1) = 2n^2 \qquad k(c_1 - b_1) = G^2$$

or

$$c_1 - a_1 = (2n^2/k) \qquad c_1 - b_1 = (G^2/k).$$

Since $k > 2$ it would follow that k divides n and G, which is contrary to the fact that n is prime to G.

6. *The inradius method.* Given the inradius r, find all the triples corresponding to r which are primitive.

From (10), $r = nG$ where n and G are relatively prime, and G can only be odd.

Example: Let the inradius $r = 210$.

Then $r = nG = 2 \cdot 3 \cdot 5 \cdot 7$, and by assigning to G all of the possible combinations of the distinct odd primes we have the results shown in Table 1.

7. *The odd-leg method.* Given an odd leg a, find all the triples corresponding to a which are primitive.

From (1) $c^2 - b^2 = a^2$

TABLE 1

odd G	n	(a) odd $2nG+G^2$ $2r+G^2$	(b) even $2nG+2n^2$ $2r+2n^2$	(c) odd $2nG+G^2+2n^2$ $b+G^2$
1	$2\cdot3\cdot5\cdot7$	421	88,620	88,621
3	$2\cdot5\cdot7$	429	10,220	10,229
5	$2\cdot3\cdot7$	445	3,948	3,973
7	$2\cdot3\cdot5$	469	2,220	2,269
$3\cdot5$	$2\cdot7$	645	812	1,037
$3\cdot7$	$2\cdot5$	861	620	1,061
$5\cdot7$	$2\cdot3$	1,645	492	1,717
$3\cdot5\cdot7$	2	11,445	428	11,453

From (6) $c-b=g=G^2$

Divide $c+b=(a/G)^2$

From (6) $c-b=G^2$

Subtract $2b=(a/G)^2-G^2$

(14) $$b=\tfrac{1}{2}\left[(a/G)^2-G^2\right].$$

Example: Let the odd leg $a=315=3^2\cdot5\cdot7$.

From (14) it follows that G, which is always odd, must be a factor of a, and by assigning to G all of the possible combinations of the distinct primes, which are all odd since a is odd, we have the results shown in Table 2.

TABLE 2

odd G	(b) even $\frac{1}{2}[(315/G)^2-G^2]$	(c) odd $2nG-G^2-2n^2$ $b-G^2$
1	49,612	49,613
5	1,972	1,997
7	988	1,037
3^2	572	653

8. *The even-leg method.* Given the even leg b, find all the triples corresponding to b which are primitive.

Using equations (1) and (7) in the same manner as (1) and (6) were used in Section 7, we have

(15) $$a=(b/2n)^2-n^2.$$

Since, in (15), b is divisible by $2n$, let $b=2b_1$ so that

(16) $$a=(b_1/n)^2-n^2.$$

We will now show that b_1 must also be even, for if it is odd and divisible by n, then n must be odd and we have

$$a=(\text{odd})^2-(\text{odd})^2=\text{even}.$$

This conclusion contradicts the fact that a is odd.

Now, from (12), $b=2nG+2n^2$, so that $b>2n^2$, and since $b=2b_1$, it follows that $2b_1>2n^2$ or $\sqrt{b_1}>n$, i.e., $n<\sqrt{b_1}$.

Example: Let the even leg $b=6{,}120$.

$b_1=(b/2)=3{,}060=2^2\cdot5\cdot3^2\cdot17$, which satisfies the requirement that b_1 be even.

$n<\sqrt{b_1}=\sqrt{3{,}060}=56$ approx.

From (16), n must be a factor of b_1, and by assigning to n all the possible combinations of the distinct primes, whose products are less than 56, we have the results shown in Table 3.

TABLE 3

n	(a) odd $(3060/n)^2-n^2$	(c) odd $2nG+G^2+2n^2$ $a+2n^2$
1	9,363,599	9,363,601
2^2	585,209	585,241
5	374,519	374,569
3^2	115,519	115,681
17	32,111	32,689
$2^2\cdot5$	23,009	23,809
$2^2\cdot3^2$	5,929	8,521
$5\cdot3^2$	2,599	6,649

9. *The grouping of primitive Pythagorean triples.* Every primitive triple belongs to a specific Group G. The values of a, b, c, and r for the nth triple belonging to any group can be expressed in terms of G and n, where G is odd and G and n are relatively prime.

These formulas are:

(10) $$r=nG.$$

(11) $$a=2nG+G^2. \quad \text{(Odd)}$$

(12) $$b=2nG+2n^2. \quad \text{(Even)}$$

(13) $$C=2nG+G^2+2n^2. \quad \text{(Odd)}$$

Table 4 shows the groupings of all primitive Pythagorean triples.

TABLE 4

(G) Group No. (odd)	(n) Triple No.	(g) $\frac{c-b}{G^2}$ (odd)	(d) $\frac{c-a}{2n^2}$ (even)	(r) $\frac{(a+b-c)/2}{nG}$	(a) $\frac{2r+g}{2nG+G^2}$ (odd)	(b) $\frac{2r+d}{2nG+2n^2}$ (even)	(c) $\frac{2r+g+d}{2nG+G^2+2n^2}$ (odd)
1	1	$1^2=1$	2	1	3	4	5
	2		8	2	5	12	13
	3		18	3	7	24	25
	4		32	4	9	40	41
	5		50	5	11	60	61
	6		72	6	13	84	85
	etc.						
3	1	$3^2=9$	2	3	15	8	17
	2		8	6	21	20	29
	3		*	*	*	*	*
	4		32	12	33	56	65
	5		50	15	39	80	89
	6		*	*	*	*	*
	etc.						
5	1	$5^2=25$	2	5	35	12	37
	2		8	10	45	28	53
	3		18	15	55	48	73
	4		32	20	65	72	97
	5		*	*	*	*	*
	6		72	30	85	132	157
	etc.						

* n and G contain a common factor, i.e., they are not relatively prime.

Have you read?

LANGFORD, W. J. "Secondary School Mathematics, an International Survey," *Mathematical Gazette*, October 1958, pp. 177–193.

This is Mr. Langford's presidential address to the Mathematical Association of Great Britain. He has attempted in this address to take a large look at the aims of secondary school mathematics in the world. He finds these aims always include the laying of a foundation of mathematical knowledge and skill, an appreciation for the exact sciences, and, often, cultural enrichment. In two-thirds of the countries, mathematics is required of all. In some, it is required only of boys.

Mr. Langford briefly tells of courses that are taught that you might not expect, such as: the theory of errors; an introduction to the principle of axioms; irrational and imaginary numbers; translation; rotation and revolution; the concepts of infinitely large, infinitely small; the solution of binomial equations of third, fourth, and sixth degree, as well as many others.

From here he presents methods of instruction and teacher preparation. The first sentence of his closing paragraph sounds familiar: "The hardest task of all will be to insure that sufficient capable, qualified and willing people are attracted into the teaching profession."—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Selection of topics from solid geometry for a one-year course in geometry

DWAIN E. SMALL, *Southern Illinois University, Carbondale, Illinois.*

If solid geometry is to be taught with plane geometry rather than as a separate subject, what topics should be retained?

THE CURRENT TREND in secondary mathematics curriculum revision to eliminate solid geometry as a separate course has placed the secondary mathematics teacher in a position of indecision. Are the topics of three-dimensional geometry as previously taught no longer needed in the secondary program? Are there topics from solid geometry that should be included in a one-year course combining two- and three-dimensional geometry?

The author in attempting to devise a course that would fuse the topics of two- and three-dimensional geometry found himself making topic selection decisions. Not wanting to be too presumptuous, the decision was made to solicit the opinions of some of the leaders in the field of mathematics education.

To compile a questionnaire of topics, six[1] of the solid geometry texts that are presently in use in our nation's schools were analyzed for content. Using the criterion of appearance in three or more texts, a list of 109 items was formed. No attempt was made to distinguish between postulates, definitions, and theorems. The topic items could be listed under the following area headings:

1. A line perpendicular to a plane
2. A line parallel to a plane
3. Two line segments equal
4. Loci
5. Two lines parallel
6. Perpendicular planes
7. Parallel planes
8. Equal dihedral angles
9. Equal and symmetric trihedral angles
10. Polyhedral angles
11. Congruent polygons
12. Congruent prisms
13. Prisms equal in volume
14. Area of prisms
15. Volume of prisms
16. Area of cylinders
17. Volume of cylinders
18. Pyramids equal in volume
19. Area of pyramids
20. Sections of pyramids
21. Volumes of pyramids
22. Sections of cones
23. Areas of cones
24. Volumes of cones
25. Great circles of sphere
26. Spherical distances

[1] H. W. Hart and Veryl Schult, *Solid Geometry* (Boston: D. C. Heath & Co., 1952), 198 pp.
Virgil S. Mallory, *New Solid Geometry* (Chicago: Benj. H. Sanborn & Co., 1943), 302 pp.
Frank M. Morgan and W. E. Breckenridge, *Solid Geometry* (New York: Houghton Mifflin Co., 1946), 330 pp.
D. P. Smith and A. I. Marino, *Solid Geometry* (Columbus: Charles E. Merrill Co., 1950), 314 pp.
R. R. Smith and J. R. Clark, *Modern-School Solid Geometry* (New York: World Book Co., 1949), 256 pp.
A. M. Welchons and W. R. Krickenberger, *Solid Geometry* (New York: Ginn and Co., 1950), 300 pp.

27. Spherical angles
28. Spherical areas
29. Volumes of spheres
30. Spherical polygons
31. Spherical areas
32. Prismatoid
33. Inclination of a line and projection

With the assistance of colleagues, fifty mathematics educators were selected and were each sent a letter explaining the study and requesting their assistance. Of the fifty contacted, forty of the educators agreed to check the list that would be forwarded to them. Only thirty-one of the returned thirty-six questionnaires were usable.

The following is an analysis of the professional experience and background of the thirty-one educators whose responses were used:

1. Five are secondary-school mathematics teachers. Ten are mathematics teachers in liberal arts colleges; two are mathematics teachers in engineering schools; fourteen are mathematics teachers in teachers colleges.
2. Eighteen have taught geometry in secondary schools.
3. Six have written geometry texts for high school students.
4. Fourteen have written college texts in mathematics.
5. Twenty-five are members of the Mathematical Association of America.
6. Twenty-nine are members of the National Council of Teachers of Mathematics.

The participants were asked to indicate their opinion on each of the 109 items by checking each item in one of four ways:

1. Formal Proof. A formal deductive proof should be given by the instructor or student.
2. Informal Proof. A discussion of the proof and its consequences should be given by the instructor.
3. Postulate. Should be given as a statement of fact without proof.
4. Omit. Should not be included in the one-year course.

Since the questionnaire was lengthy, a complete tabulation of responses has not been included in this report.[2] Let it suffice to say that some mathematics educators are reluctant to give up the topics of solid geometry in the secondary curriculum. Several of the participants would have more than 50 per cent of the 109 items proved formally in a one-year geometry course. Since this would seem impractical, both from the mathematical and time-efficiency standpoint, the author chose different levels of agreement for rejection or acceptance of topics.

The 50 per cent level was chosen for the complete rejection of items. If 50 per cent of the participants checked "Omit," the item was rejected. Of the 109 items, 14 were checked "Omit" by 50 per cent or more of the participants. The topics covered by these items were:

1. Congruence of prisms
2. Spherical triangle
3. Volume of prismatoid

Since three of the four possibilities for checking were in the category of acceptance, a 75 per cent level of acceptance was chosen for inclusion of a topic. There were 58 items selected by 75 per cent or more of the participants for inclusion. The topics accepted for inclusion were:

1. A line perpendicular to a plane
2. A line parallel to a plane
3. Two line segments equal
4. Loci
5. Two lines parallel
6. Perpendicular planes
7. Parallel planes
8. Equal dihedral angles
9. Polyhedral angles
10. Area of prisms
11. Volume of prisms
12. Area of cylinders
13. Volume of pyramids
14. Volume of cylinders
15. Area of pyramids
16. Sections of cones
17. Area of cones
18. Volume of cones
19. Great circles of sphere
20. Spherical distances

[2] A list of the 109 items with a complete tabulation of the questionnaire responses may be obtained by sending a request to the Educational Research Bureau, Southern Illinois University, Carbondale, Illinois.

21. Spherical angles
22. Spherical areas
23. Volume of spheres
24. Inclination of a line and projection

If there is any merit in the information gained from this study, it rests in the guidance for selection and treatment of topics from solid geometry to be included in a one-year geometry course. The individual group of teachers who are revising the mathematics curriculum will ultimately make the decisions on inclusion of topics and whether they will be proved or postulated.

What's new?

BOOKS

SECONDARY

Industrial Mathematics, Matthias A. Thies. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1959. Cloth, ix + 342 pp., $6.75.

Intermediate Algebra (rev. ed.), Jack R. Britton and L. Clifton Snively. New York: Rinehart and Company, Inc., 1959. Cloth, x + 353 pp. $3.00.

The Modern Slide Rule, Stefan Rudolf. New York: The William-Frederick Press, 1959. Paper, 58 pp., $5.00.

Plane Trigonometry, Nathan O. Niles. New York: John Wiley and Sons, Inc., 1959. Cloth, xi + 234 pp., $3.95.

COLLEGE

Elements of Calculus and Analytic Geometry, George B. Thomas, Jr. Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1959. Cloth, x + 580 pp., $7.50.

Finite Mathematical Structures, John G. Kemeny, Hazelton Mirkel, J. Laurie Snell, and Gerald L. Thompson. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1959. Cloth, xi + 487 pp., $7.95.

Fundamentals of Freshman Mathematics, C. B. Allendoerfer and C. O. Oakley. New York: McGraw-Hill Book Company, Inc., 1959. Cloth, xiii + 475 pp., $6.50.

Introduction to Analysis, Norman B. Haaser, Joseph P. LaSalle, and Joseph A. Sullivan. Boston: Ginn and Company, 1959. Cloth, xiv + 688 pp., + xxxi, $8.50.

An Introduction to Mathematics, Donald W. Western and Vincent H. Haag. New York: Henry Holt and Company, Inc., 1959. Cloth, xi + 580 pp., $7.50.

Introductory College Mathematics, Thomas L. Wade. New York: John Wiley and Sons. Inc., 1959. Cloth, xiv + 319 pp., $5.50.

A Modern Introduction to College Mathematics, Israel H. Rose. New York: John Wiley and Sons, Inc., 1959. Cloth, xxi + 529 pp., $6.50.

MISCELLANEOUS

The Canterbury Puzzles (4th ed.), H. E. Dudeny. New York: Dover Publications, Inc., 1958. Paper, 255 pp., $1.25.

The Math Entertainer, Philip Heaford. New York: Emerson Books, Inc., 1959. Cloth, 176 pp., $2.95.

Mathematics Dictionary (student's edition, 2nd ed.), Glenn James and Robert G. James. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1959. 474 pp., $10.00.

Pillow Problems and a Tangled Tale, Lewis Carroll. New York: Dover Publications, Inc., 1958. Paper, xx + 152 pp., $1.50.

Symbolic Logic and the Game of Logic, Lewis Carroll. New York: Dover Publications, Inc., 1958. Paper, xxxi + 96 pp., $1.50.

BOOKLETS

Mathematical Needs of Prospective Students (rev. ed.), College of Engineering, University of Illinois. University of Illinois Bulletin, Vol. 56, No. 51, 1959. Paper, 28 pp., $.50.

Mathematics for the Academically Talented Student, Julius H. Hlavaty, editor. Washington, D. C.: National Education Association and National Council of Teachers of Mathematics, 1959. Paper, 48 pp., $.60.

Teaching Mathematics in Secondary Schools, Ministry of Education Pamphlet No. 36. London: Her Majesty's Stationery Office, 1958. Paper, vii + 154 pp., $1.16.

Reason vs. reflex in the teaching of mathematics

WILLIAM G. LISTER, *State University of New York, Oyster Bay, New York.*

"A language in which mathematical statements can be made concisely and clearly does not have to be invented. There is a very satisfactory one in use among those whose business it is to create and communicate new mathematics, but it is largely ignored or misinterpreted in elementary work."

IN THE PAST FIVE YEARS signs of dissatisfaction with the state of mathematical education in the secondary school have appeared with increasing frequency and intensity. At the present time noteworthy activity in curricular renovation and reorganization is under way locally at the departmental and school system level and nationally through the College Entrance Examination Board Commission on Mathematics and the School Mathematics Study Committee now beginning work under the administration of the mathematics department at Yale. For the first time in far too long the topics of instruction are receiving a serious examination by many groups of influence, authority, and competence.

But topical considerations lead directly to reflections on the potential contributions of secondary mathematics in the educational process, on its present relation to applications, on its contribution to strong mathematical foundations for future work, and on its role in the development of intellectual maturity, precision, and analytic power. It is my conviction that attention to the selection of topics is *not* the most effective approach to the problem of constructing a program adequately realizing such aims. Much more to the point, although certainly not independent, are considerations deriving directly from the central aims of mathematical instruction. We must reflect on our present understanding of the fundamental nature of the subject. How is it created? How are the ideas to be expressed and how understood? We must ask such questions now because, insofar as it represents implicit answers to them, current practice exhibits a faulty and inadequate understanding.

Today's (and yesterday's) approach is mistaken in treating mathematics as if it is similar to the Law in being enacted or decreed, in being loosely based on general principles, and "learned" by developing a mental encyclopedia of fragments. It is mistaken in operating as if mathematical ideas are of such a different kind from others that they can—or must—be transmitted by means other than coherent statements which are grammatically and logically complete. Without acknowledging it explicitly, we have been insisting in practice that the primary aims of secondary instruction in mathematics are (1) the development of appropriate conditioned reflexes in the student by means of drill in accurate response to non-rational procedural directions, and (2) the accumulation of certain arithmetic and geometric factual material which is regarded as static and discrete, thoroughly isolated from any environment within which it can be comprehended. The

reflex responses are pretentiously called skills and the so-called facts are often false or incoherent.

These practices reflect disparaging attitudes which greatly diminish the stature of mathematics. Behind them apparently is a belief that algebra and geometry can be presented as broadly practical skills, such as reading, writing, typewriting, and automobile driving. These mathematical skills are supposed to be valuable generally as mental disciplines and narrowly as necessary techniques for the future engineer and scientist. As we now treat these subjects neither of the claims for them will, I think, stand examination. Of the students who do not continue scientific studies in college, very few will be called upon to use high school mathematics explicitly and even fewer would be able to do so. As for those whose careers will be in science or technology, generally they will either make very trivial use of mathematics and require little more than a vocabulary and a few books to provide answers to standard problems, or they will need to possess a thorough control of certain branches and to reach a significant level of mathematical maturity. For routine uses no basic knowledge at all is needed, while for significant applications ingenuity, flexibility, and an understanding of the efficacy and limitations of many techniques are essential. As we all know, a school training in routine will be forgotten anyway.

Now, many teachers and ex-algebra students have, in fact, been forced to the conclusion that the "useful skill" orientation of mathematics beyond arithmetic is not defensible. When pressed on this point, a school or college teacher often shifts his ground and defends the work he requires as good training in rigorous thinking—fine stuff to sharpen the mind. Nonsense! When mathematics is reduced to a language (with a highly irregular grammar) or an almost manual skill, it has no more to recommend it in this regard than Esperanto and perhaps less than typing. In this sense the criticisms made by school administrators during the last generation were justified. They did not know what the subject *might* be, and if the colleges had not insisted, often for prestige or from inertia, that some mathematics be presented for admission, we should undoubtedly have seen a much greater decrease in its quantitative importance in the secondary curriculum.

One sign that our representation of mathematics is badly out of joint is the popular reaction to mention of contemporary mathematical activity. When experts assert that mathematics is important in scientific development, the educated public must *believe*, but it cannot *comprehend*. Whether mathematics in school represented a psychic trauma or a pleasant but superficial game, the layman has difficulty in imagining its actual use. He is at a total loss in imagining the nature of the mathematics that can be used. Since he necessarily thinks of mathematics in terms of his own experience, he is badly confused, easily duped, and suspicious of fraud. In the popular image a mathematician is a man in a green eyeshade summing immense columns, doggedly extracting square roots, and solving new equations. If the layman is intelligent and concerned, he has a right to feel doubly cheated—first, because he has never been shown the relation of mathematics to the sciences and given a glimpse of it at work and, second, because the subject as he knew it could not possibly be the one which is pursued for its own sake and is often almost as far from physics as nonrepresentational drawing. Of all the intellectual disciplines, mathematics is distinguished in being conducted completely out of public sight. This makes for an ignorance and indifference which tolerate abuses. Although such a state of affairs can at times seem desirable to the practitioners, surely the wisdom of creating a body of educated laymen who understand the nature of the subject today, the way it develops, and its role in the study of natural phenomena is not arguable.

In very general terms I have indicated what seem to me reasonable aims and proper values in the study of mathematics by intelligent secondary school students. Only in attempting to implement these lofty generalities does the arduousness of the task become clear. There are many aspects of the somewhat more specific proposals to follow that are capable of arousing controversy. Any program which asks more of the teacher will meet resistance. The program will be difficult to give an initial impulse, difficult to keep under way, and defeated if it ever achieves a static, finished form. I would approach the problem by singling out three or four closely related elements of mathematics, each to be consciously developed through every course of instruction.

First, an adequate language for the transmission of mathematical thought must be developed. Second, the logical basis of mathematical validity must be made explicit and the ABC's of informal logic must become a part of every student's equipment. Third, the content of mathematics should grow out of a very few notions whose existence and operational properties are agreed upon or formalized in axiom systems. Fourth, the difference between the mental constructions of mathematics and any "real" counterparts, the sense in which mathematics applies to phenomena of experience, must be presented clearly, constantly, and in a steadily more mature form as the student grows. These four elements provide a growing structure within which all mathematics at all levels can be studied and in terms of which foreseeable developments can be expressed. At the graduate school level these ingredients are fairly well imbedded in standard practices, but since it is often difficult to find traces of them in school and college, my remarks are aimed in that direction.

A language in which mathematical statements can be made concisely and clearly does not have to be invented. There is a very satisfactory one in use among those whose business it is to create and communicate new mathematics, but it is largely ignored or misinterpreted in elementary work. It is not a separate language but rather an adaptation of common modes of expression to the types of multiple assertions made in mathematics. The adaptation is accomplished by introducing a new form. The form has technical names, but it is based on the replacement of one or more designations occurring in a sentence by "blanks." Symbols are used to mark the blanks, usually letters. The result is of course not an assertion at all but can be converted into one by asserting the truth of certain statements which result from replacing the blanks by appropriate designations. For example,

$$(5-1)(5+1)=25-1$$

is an assertion, a correct one as a matter of fact. But

$$(x-1)(x+1)=x^2-1$$

is not a statement any more than

x is the smallest of the United States.

In each case x is understood to mark a blank to be filled in. Any symbol not causing confusion would do equally well. However,

$$\text{for every real number } x,\quad (x-1)(x+1)=x^2-1$$

is an assertion. It asserts the conjunction of all statements formed by replacing x by a real number throughout; that is, it asserts that every such substitution produces a true statement. Many different assertions can be constructed in this same general way from a suitable expression with blanks.

$$2x+3=5$$

could be turned into:

1. There is at least one real number x such that $2x+3=5$.
2. There is exactly one real number x such that $2x+3=5$.
3. There is no real number x for which $2x+3=5$.

All these are statements in the logical sense. Their operational meaning should be clear; for example, 3 asserts the conjunction of the negations of all statements formed by substituting real numbers for x in "$2x+3=5$." Of the three, 1 and 2 are true while 3 is false.

The same mode of expression is also valuable in forming collective designations; that is, in specifying entities which are aggregates or, to use the standard mathematical term, sets. Thus

the set of all real numbers x such that
$$(x-1)^2>4$$

denotes such an object; namely, the conceptual collection composed of all real numbers whose names when substituted for x in "$(x-1)^2>4$" produce true statements. Since the objects of mathematical investigation can practically all be regarded as sets with a resulting gain in unity and simplicity, this type of verbal construction is fundamental.

I am convinced that attention to the development of the notion of substitution for blanks provides the only addition to ordinary verbal forms needed to permit concise and accurate mathematical communication. The idea would seem to be simple enough for early understanding and accurate use, but in the exposition of algebra and calculus corruptions and truncations of this technique have become so orthodox that teachers, writers, and students alike are conditioned into insensitivity to the distinction between sense and nonsense. To discuss the linguistic absurdities which replace clear statement in standard text treatments would require many pages.

A thoroughgoing use of the verbal forms indicated above has the additional advantage of displaying clearly the logical character of any assertion. Indeed, the common practice of excising all components of an argument except the equations appearing in it leaves the impression that no argument is required or even appropriate. For example, the sequence of symbols

$$x^2=4$$
$$x=2, \qquad x=-2,$$

if it is to be made intelligible, can be taken as an abbreviation of

for all real numbers x, $x^2=4$ if and only if
$$x=2 \text{ or } x=-2.$$

But there are other assertions of which the given symbols might be taken as a contraction, and only if such an argument is written completely is it possible to say what the logical aspects of the process of equation solving are and whether a given process is in fact valid. Only by making the various mathematical procedures and processes fully explicit can their validity be easily examined. At the same time the student who works under such discipline improves both his verbal and rational power.

One important problem in this connection is to determine the stage of maturity at which the student can verbalize his reasoning. Some aspects of the work of Beberman and others at Illinois bear on this problem, while the program as a whole affords an unusual example of an attempt to test and implement some of the principles advocated in this article. There is an account of Beberman's program in an article by E. P. Rosenbaum in the May 1958 *Scientific American*. Whatever details are uncovered in the next few years it seems certain that consciousness of logical concepts and the ability to express them develop gradually. It is my guess that well before the level of admission to college a working effectiveness can be reached and that if the right start is not made during this period of readiness, the process becomes much more difficult because mental habits are too firmly set.

There is another reason for implicit emphasis on logic in the course of instruction in mathematics. It seems impossible to teach the correct application of logic by presenting the subject formally and abstractly from the start. A substantial experience in the application of logical principles, principles resting on an appeal to common sense, seems to be prerequisite. Since mathematics is the only coherent

subject field wherein the concepts are defined sharply enough to permit the exclusive use of deductive argument and in which there is subject matter logically and representationally rich enough to give the results significance, it provides an ideal vehicle for the development of a working familiarity with logic. By emphasizing this aspect of the subject, we can reasonably expect to develop logical sensitivity and verbal precision as a part of the student's intellectual equipment.

Building mathematics on a very few comprehensive and basic notions is crucial for the clarification and simplification of its concepts and vocabulary. As I have already suggested, the consistent use of a suitable mode of expression leads naturally to the consideration of *sets*. Furthermore, the algebra of sets, a very simple and unsophisticated subject, could be blended into the curriculum at a very early stage. Thereafter it could be used in such a way that the objects one studies in the algebra of the real number system, in plane analytic geometry, in the study of relations and functions, are all sets of some sort. If geometry is to be developed from separate axioms (the synthetic point of view), the notion will be less prominent but is still useful. During the past few years I have had the chance to teach a two-year calculus sequence developed by Professors H. Federer and B. Jonsson. It consistently and, I think, successfully employs sets as the common conceptual currency. Any presentation of probability theory or statistical analysis must either do the same (see, for example, Feller's *Introduction to Probability Theory and Its Applications*) or obscure the notions on which these subjects are now based. As a matter of fact, if one wishes, almost all definitions can be regarded as the singling out of certain sets or types of sets.

There is, of course, no overwhelming reason why definitions should be framed in terms of sets unless it is natural and direct to do so. However, an important contribution of any revised approach to mathematics would be the elimination of unnecessary proper nouns. If, instead of introducing every bit of the terminological accumulation of generations (some of it incomplete, some inaccurate), we would leave out all terms which do not definitely improve the efficiency of definition and communication in subsequent work, an entirely artificial difficulty in learning mathematics would thereby be removed.

The fourth element in a new approach to mathematical instruction is the relation of mathematics to "real" phenomena; that is, to nature. This is *not* strictly a part of mathematics and, almost as a consequence of this fact, is a subtle, adult topic understood only as maturity both in the process of abstraction and in the analysis of nature develops. The modern attitude toward this aspect of mathematics has been achieved only within the past hundred years, and we must expect students to repeat the historical sequence of attitudes to some extent. Nevertheless, by the time an axiomatic study of plane geometry is attempted, some attack must be made on the problem; yet I know of no secondary level text which makes any explicit attempt to do so. For the most part geometry seems to be regarded as the direct study of reality or of some idealization of reality. This is a primitive and restrictive point of view which must gradually be abandoned. In the process no textbook is any sort of substitute for a teacher whose understanding is not ten pages but a century beyond that of his students. Probably the student can begin in school to achieve some comprehension of the nature of the number concept and of the sense in which the rational, real, and complex numbers "exist." Comprehension of the complete neutrality of axiom systems as far as their representation goes, and of the inherent difference between mathematical knowledge and physical knowledge, must also proceed gradually, but it should begin as early as possible. This sort of comprehension adds an additional dimension to the study of mathematics and it should stimulate reflections

beyond the realm of mathematics. Programs like Beberman's could provide information about the level of sophistication in this matter achievable at each stage of student development.

In emphasizing the necessity for replacing the conditioned reflex approach to mathematics by cultivating reliance on the individual's rational and imaginative senses. I have ignored the role of any particular topic. I do not mean to imply that the choice of topics is a matter of indifference. The desired change of attitude would necessarily transmute certain aspects of every topic. Apart from such natural metamorphoses there are a few subjective criteria which I would keep in mind in constructing the curriculum beyond arithmetic.

1. Prefer what is fundamental to further progress in mathematics. This seems to me to imply early and systematic use of sets, gradual development of the language and logic necessary for thought and expression, thorough familiarity with the algebra of the real number system, with its relation to the line, and with such other elementary geometric notions as *distance, circle, angle,* and *rigid motion.* The concepts of *relation* and *function* and the pictorial representation of them in the plane are also fundamental.

2. Choose topics which are rich without being too sophisticated or complex. Prefer that which seems clear, beautiful, or remarkable. Discuss topics which the students are not equipped to penetrate or for which the required time for significant study is unavailable only when the necessity for using some of the resulting techniques is immediate and inescapable. This sort of dilemma has been faced commonly in the case of students who are beginning work in physics or engineering science.

3. Give consideration to topics which have important applications outside of mathematics, in particular those whose use almost everyone is expected to understand or evaluate. The mathematics of finance and the basis of statistical analysis in probability theory come immediately to mind. Of course, reference to criteria such as these three may do no more than focus attention on disagreements about which programs exhibit the desired qualities.

One final word by way of emphasis. The accomplishments of any reorientation of mathematical instruction must be measured in terms of student development alone. It is the practice of some authors and teachers to present the subject on one level but to require the student to respond only on a much more superficial or mechanical level. Such pretension is worse than useless. To learn to write accurately in mathematics, a student must be expected to do so on each response. To learn to reason correctly and fluently, he must first work constantly at the construction of simple arguments. This is hard work, but it is productive work. Its difficulties are real but not insuperable and not arbitrary. The personal satisfaction of developing the capacity for discovery, of being able to judge and even to create mathematics oneself, can, and perhaps must, provide motivation to face the next demand.

"The whole of mathematics consists in the organization of a series of aids to the imagination in the process of reasoning."—*A. N. Whitehead, in* Universal Algebra, *Cambridge, 1898, p. 12.*

How to use the perpetual calendar

DONALD JOE KESSLER, *Aldine Senior High School, Houston, Texas.*[1]

Why does it work?

THE CALENDAR IS divided into three parts. Table 1–2 is that of the year. The top part of Table 3 contains the months, and the bottom part of Table 3 contains the days of the week and days of the month. To find the day of the week that any date falls on, for example, December 7, 1941, one must first find the number for that year. The first two digits of the year are found in Table 1. The last two digits are found in the bottom section of Table 2. Note that the columns in which these numbers lie cross (one horizontally and one vertically) in the top half of Table 2. Here one finds the year number. For 1941 it is 2. Now to find the number for the month of the year, use the top part of Table 3. Using the number just found, 2, go down the column under this number until the month row is crossed. For December it will be 1. Locate this same number in the section of days of the week found in the bottom part of Table 3. This number is used to find the days of the month which are in columns to the right of the days of the week. In the example, one would use the set of days of the week opposite 1 which begins with "Mo" for Monday. Using this set, we find that the seventh falls on a Sunday.

[1] Submitted by his teacher, Bertha P. Jacobs, Houston, Texas.

PERPETUAL CALENDAR

TABLE 3

	1	2	3	4	5	6	0
Jan Oct	2	3	4	5	6	0	1
Feb Mar Nov	5	6	0	1	2	3	4
Apr July	1	2	3	4	5	6	0
May	3	4	5	6	0	1	2
June	6	0	1	2	3	4	5
Aug	4	5	6	0	1	2	3
Sept Dec	0	1	2	3	4	5	6
0	Su	Mo	Tu	We	Th	Fr	Sa
1	Mo	Tu	We	Th	Fr	Sa	Su
2	Tu	We	Th	Fr	Sa	Su	Mo
3	We	Th	Fr	Sa	Su	Mo	Tu
4	Th	Fr	Sa	Su	Mo	Tu	We
5	Fr	Sa	Su	Mo	Tu	We	Th
6	Sa	Su	Mo	Tu	We	Th	Fr
	1	2	3	4	5	6	7
	8	9	10	11	12	13	14
	15	16	17	18	19	20	21
	22	23	24	25	26	27	28
	29	30	31				

TABLE 1

16	20	24
17	21	25
18	22	26
19	23	27

TABLE 2

6	0	1	2	3	4	5
4	5	6	0	1	2	3
2	3	4	5	6	0	1
0	1	2	3	4	5	6
00	01	02	03	XX	04	05
06	07	XX	08	09	10	11
XX	12	13	14	15	XX	16
17	18	19	XX	20	21	22
23	XX	24	25	26	27	XX
28	29	30	31	XX	32	33
34	35	XX	36	37	38	39
XX	40	41	42	43	XX	44
45	46	47	XX	48	49	50
51	XX	52	53	54	55	XX
56	57	58	59	XX	60	61
62	63	XX	64	65	66	67
XX	68	69	70	71	XX	72
73	74	75	XX	76	77	78
79	XX	80	81	82	83	XX
84	85	86	87	XX	88	89
90	91	XX	92	93	94	95
XX	96	97	98	99		

If the year is a leap year, one *must* subtract the number 1 from the number found for the month, if the month is *January* or *February*. If the month is *not* January or February, *do not* subtract anything.

I got the principle for this calendar from THE MATHEMATICS TEACHER.[2] The calendar can easily be made small enough to be put in a wallet, which I have already done. Although the years range from 1600 to 2799, one can extend the range of the calendar by noting the way the first two digits of the year are arranged. Any that should be added will follow the same form. For example, it would look like this:

12	16	20	24	28
13	17	21	25	29
14	18	22	26	30
15	19	23	27	31

[2] Rev. Brother Leo, O.S.F., "A Mental Calendar," THE MATHEMATICS TEACHER, L (October 1957), 438–39.

"The role of the teacher is vital: curricular change must be accompanied by effective, meaningful teaching, directed toward the development of mathematical power and understanding."—"Program for college preparatory mathematics," *Report of the Commission on Mathematics, College Entrance Examination Board, New York, 1959, p. 15.*

Letter to the editor

Dear Editor:

The following poem was written by Lucile B. Canfield for the National Science Foundation Science and Mathematics Institute held at Montana State University, summer 1959. Mrs. Canfield teaches science and mathematics at Richey High School, Richey, Montana. Some of her hobbies are Montana history, geology and rocks, junior rodeos, and horse shows.

MODERN MATH

I've taught the kids for 20 years
That 6 plus 1 is seven
And also in the standard way
That 10 and 1 are eleven.

But 6 and 1 are zero now
And 10 plus 1 is four,
Of course that's counting circular
With modulus no more.

And U is up and U is down
No cow's hide does it brand—
But intersection unions
In mathematics land.

And $a * b$ is binary
By golly they can prove it,
Associative, commutative,
No abelians disprove it.

And then there're groups of objects
Whose arrangements are absurd
For the 1st is in the 2nd's place
And the 2nd in the 3rd.

Now all this may be modern
And used in math to boot,
But I'm getting like the herder
Who's as crazy as a coot.

Submitted by,
MRS. E. R. SIEVERS
Missoula, Montana

Notes on a "spider" nomograph

LOWELL T. VAN TASSEL, *Herbert Hoover Senior High School, San Diego, California.*

THE NOMOGRAPH, or alignment chart as it is sometimes called, shown in Figure 1 can be used to solve problems involving the reciprocal sum equations from the physics of optics, electricity, and from the rate problems of algebra. The so-called lens equation of optics states that the sum of the reciprocal of the object distance plus the reciprocal of the image distance is equal to the reciprocal of the focal length of the lens involved. A general form of all such equations is given by equation (1):

$$\frac{1}{A}+\frac{1}{B}=\frac{1}{C}. \tag{1}$$

Figure 1 is the nomographic expression of equation (1).

The units in Figure 1 are of arbitrary equal length, and all three scales are congruent. The angles between legs are each 60°. The zero point (O, origin) lies at the vertex, or point of concurrency.[1]

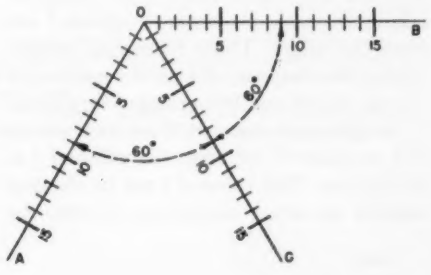
The purpose of this paper is twofold: (i) to illustrate by the use of the alignment chart, Figure 1, how the three parameters of equation (1) may be varied; and (ii) to develop a "spider" nomograph for the solution of equation (2), a general form of (1), having more parameters (degrees of freedom) than (1). In the case of the "spider," it is to be noted that one need never estimate any intermediate values in the process of solving equation (2) on the nomograph. The writer will show that a "spider" may have several legs amputated and yet function in an adequate fashion, much like its arachnidan namesake.

The construction of the "spider" is an excellent exercise for an elementary geometry class. Only the classical instruments, compass and straightedge, are used to make and exploit the "spider" of Figure 2. The application of Figure 1 or Figure 2 in solving problems in the physics class should create considerable student interest. Almost any algebra book will serve as a source of rate problems whose answers can be readily estimated by using alignment chart technique.

The basic construction of Figure 2 is essentially that of constructing a regular hexagon, or the familiar six-petal flower design often drawn in art classes. Here, $OP_1=OP_2=P_1P_2$. A geometry class, depending upon its level of attainment, should be able to prove certain simple facts concerning Figure 2; e.g., triangle OP_1P_2 is isosceles and equilateral and arc $P_1P_2=60°$.

A "spider body" of unit radius may be drawn, centered at O and having OK as radius. In Figure 2, all three cross-diame-

Figure 1



[1] See "A Nomograph Based on Soap Bubble Geometry," *School Science and Mathematics*, LVII, Number 9 (December 1957), 689–93, for an elementary proof using the law of sines. Many books on nomography give the result in more general form: $1/f(x)+1/g(x)=1/h(x)$.

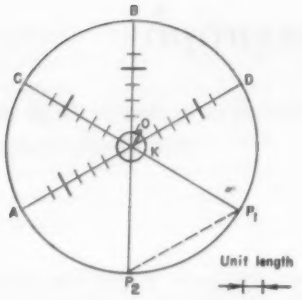


Figure 2

ters should be done in dark pencil (or inked in). A convenient unit length (say 1 cm) is chosen, and a scale is marked off and numbered 1, 2, 3, · · · , on every leg. Figure 2 may be drawn on any square piece of paper, say $12'' \times 12''$.

For what follows, we need use only a semicircle of Figure 2. Our first announced purpose (i) was to exhibit the use of Figure 2 (Figure 1 is implicit in Figure 2) in the solution of equation (1). The teacher here may bring out the lens formula from optics, the work-rate problems of algebra, and the R-C circuits of electricity,[2] each of which problems involves essentially equation (1).

Let us consider a typical lens problem from physics, and its solution on the nomograph of Figure 1. Assume that the following information is given: the distance from a lens of focal length f (f to be determined) to an object is 8 cm; the distance from the same lens to the image produced by the object is 4 cm. We ask: What is the required value of f that will produce this object-image situation? Using Figure 2, on scale A locate a point 8 cm from the origin. On scale B, locate the point 4 cm from the origin. These represent, respectively, the distance of lens of focal length f, from object and from image. Now using a straightedge, connect these two points and on scale C estimate the value of f to be 2.67 cm. This value of f can be checked against an exact numerical solution by solving equation (1), using the given data.

A variation of this problem is to consider equation (1) as involving independent parameters. Let us arbitrarily fix C to be 2.67 cm, the required (for some purpose) focal length. Then it is possible to find all (positive) object and image distances that combine reciprocally to yield the required f of 2.67. To do this easily, we insert a map-user's or common pin into scale C at 2.67, perpendicular to the plane of the alignment chart. This pin then is used as a fulcrum point for the straightedge to be rotated about, the rotation being in the plane of the paper. Noting distances on scales A and B simultaneously, we read among other possible combinations, $A=8$, $B=4$. We observe that the symmetric opposite pair $A=4$, $B=8$ also appears. (In a science class, this should be exhibited optically.)

The student should be permitted to experiment, to locate some other values, and then to verify arithmetically his nomographic solutions. In some mathematics classes, the teacher might introduce the concept of infinity by allowing A to become large without limit; then the straightedge is nearly parallel to the OA axis. The ramifications of non-Euclidean geometry here might be mentioned in exceptional classes.

The development of the second point (ii) announced above, namely the generalization of equation (1) and full exploitation of Figure 2, follows. Let us consider a modified form of equation (1):

$$(2) \qquad \frac{1}{A}+\frac{1}{B}+\frac{1}{D}+\frac{1}{E}+\frac{1}{F}=\frac{1}{C}\,.$$

A physical analog of this situation occurs in electricity where the experimenter has only low wattage resistors available and is trying to dissipate as much heat as possible, usually subject to resistance C as fixed (see Fig. 3). For our purposes here let us make the following circuit: We wish to combine five 10 watt, 100 ohm resistance elements in parallel; by computation

[2] *Ibid.*

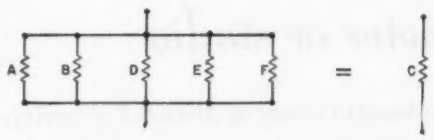


Figure 3

in (2), we calculate the effective resistance, C, to be 20 ohms. However, now our circuit (Fig. 3) will dissipate 50 watts instead of 10. Thus: five 10 watt, 100 ohm resistance elements in parallel would be the electrical equivalent of a *single* 50 watt, 20 ohm resistance. Note that the wattage (here taken as equivalent to heat-dissipating ability) is additive in the usual sense, but the resistance in ohms is not.

Let us use a large ($100'' \times 50''$) semicircle copy of Figure 2, involving legs A, B, C, and D, to solve equation (2) nomographically.[3] Although in our discussion here, we will for operational clarity estimate *intermediate values* for partial compositions of the left-hand side of (2), it must be noted that in the actual (physical) manipulation on the nomograph, *no* intermediate values need be read. Map-user's pins are used to "pin-prick" locations.

We will use the (simple) data from our previous problem. *Given:* $A = 100 = B = D = E = F$. *To find: C. Procedure:* Using $A = 100$ ohms, $B = 100$ ohms, locate their composite reciprocal sum on scale C[4] (we estimate 50 ohms). Now "add" this value (50 ohms) to 100 ohms on D. On B, read the value of this last composition to be $33\frac{1}{3}$. Now "add" $33\frac{1}{3}$ ohms on B to 100 ohms ($= E$) located on scale A. On scale C, read 25 ohms. "Add" this 25 on scale C to 100 ohms ($= F$) on scale D, and on scale B[5] read the final result, 20 ohms. This agrees with our answer by computation.[6]

The curious geometry student should study the variation of parameters in (2) as he did for (1). The really enterprising statistician might investigate this nomographic method, using the "half-spider" of Figure 2 with applied data, to see how errors accumulate in the nomographic solution of (2). Note that although a "complete spider" could be used, the "half-spider" works just as well.

Nomographs are easy to construct[7] and even easier to use. It is unfortunate that students are often ignorant of this branch of graphics until well into college studies. For the teacher who wants material on nomography, this writer has found a chapter in Jerome S. Meyer's *Fun with Mathematics*[8] to be well suited for construction, use, and understanding by the better secondary-school student. A suitably constructed (this usually means cleverly drawn) nomograph can often exhibit essential analytical details of an algebraic expression that the analytic form itself may mask. For a more sophisticated treatment of the subject of nomography, one may examine any of the books of the leading technical publishers in this field.

[3] Note that if we have each leg go out to about 110 cm, we have a circle about 250 cm in diameter, or approximately 100 inches, allowing room for margin and lettering the scales. Using a scale multiplier of 10 (implicitly), one *could* use the $12'' \times 12''$ Figure 2 already constructed and only "pin-prick" the intermediate values. The final reading then, *here*, is 2. But, 2×10 (scale factor) = 20 ohms, *answer*, as before.

[4] Note that this is *not* the final C reading.

[5] Note that the *final* C reading (here, at least) comes off the B scale. The scales have been kept uniform in nomenclature; hence this result. Once the technique is mastered, no difficulty should be occasioned in reciprocal addition using even more entries than the *five* on the left-hand side of (2).

[6] Trigonometry students may verify these intermediate values as an exercise in oblique triangles.

[7] Figure 1, for example, can be made on a large bread board by placing three identical yardsticks (or meter sticks) on edge, each touching the other at 0. The middle stick bisects the 120° angle formed by either outside stick. This idea also obviates calibrating each scale.

[8] Jerome S. Meyer, *Fun with Mathematics* (Cleveland: The World Publishing Co., 1952), pp. 100 ff.

Measurement by telemeter or stadia

W. ROWLES, *Macdonald College of McGill University, Macdonald College P.O., Que., Canada.*

A measuring device is described suitable for construction by the high school student.

THE TEACHING DEVICE described by H. J. Schiff[1] and called by him a telemeter is an interesting and useful piece of equipment. It is simple to construct and offers a chance to put trigonometry to work. It has, however, one obvious limitation in that it can be used only over level ground. If it is kept in the classroom or in the school corridor, no difficulty will be met; but if one takes it outside where the ground slopes it may give very erroneous results. From the teaching point of view this is serious, because a measuring device should work over any reasonable terrain.

Suppose the instrument is set up, by chance, on a slight incline, rising 1 foot in 50 feet. Its readings will be seriously in error, and, if used on an incline that *falls* 1 foot in 50 feet, the errors will be even worse. This is shown in Table 1.

A slope of 1 in 50 is not very great; it might indeed go unnoticed. But if one set up the apparatus on a slope which was obvious, its readings would then be quite meaningless.

To avoid this difficulty a different type of device has been designed and its method of construction is described in what follows. It also is made with simple materials and it too offers teaching possibilities. It has some qualities of a surveyor's stadia but with modifications to simplify the construction.

Operation: The principle of operation is shown in Figure 1. The observer at O, at one end of the required distance, looks toward the "rod" PQ placed perpendicular to the line of sight at the other end. He adjusts two "sights" B, C so that OBP, OCQ are straight lines. The rod, PQ, is of known length, say 8 feet, while the "sights" B, C move along a scale marked on the instrument. From this scale the required distance D may be read directly.

Theory: Let $OT = D =$ the distance to be measured and let $PQ = 8.00$ ft. Let PQ and BC be $\perp$ to OT and let $PT = TQ = 4.00$ ft. Let $\angle POT = \theta$.

By similar $\triangle$'s:

$$\frac{BE}{OE} = \frac{PT}{OT} = \frac{4.00}{D}$$

$$\therefore \quad BC = 2BE = \frac{8(OE)}{D} .$$

[1] Herbert J. Schiff, "Finding Distances with the Telemeter," THE MATHEMATICS TEACHER, XLVIII (November 1955), 473–75.

TABLE 1

Instrument reading for "perfect" device calibrated on level ground	Actual distance if used on incline rising 1 in 50	Actual distance if used on incline falling 1 in 50
10.0 ft.	9.6 ft.	9.6 ft.
20.0	18.5	21.7
30.0	26.8	34.1
40.0	34.5	47.6
60.0	48.4	79.0

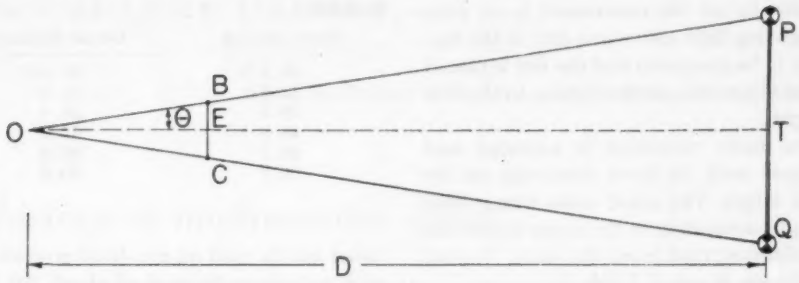


Figure 1

But

$$OE = (OB)\cos\theta.$$

$$\therefore \qquad BC = \frac{8(OB)}{D}\cos\theta.$$

Now if θ is small, $\cos\theta \doteqdot 1$, and if we allow for a possible error of 1 per cent in setting and reading the "sights" then the factor $\cos\theta$ may be neglected if θ is less than about 9°. This is equivalent to $D \doteqdot 25$ ft.

Consequently, for distances greater than 25 feet (and there is not much point in using the instrument over shorter distances), it follows that $BC = 8k/D$ where $k = OB$, a constant, which is known from the dimensions of the instrument. Thus one may calculate values of BC for values of $D = 25$, 30, 35 feet, etc., and when these are marked on the instrument (with C fixed at zero) one has a direct-reading scale.

Figure 2

Construction: It is hoped that this will be clear from the photograph. The main quadrant is made from ordinary corrugated cardboard and so is the moveable arm which carries the moving cross-wire used in sighting. The "fixed" cross-wire is glued to the main quadrant. The cross-wires are mounted on wooden vees and are made of No. 40 wire (thin nylon thread would probably be better). The sighting hole is bored with a No. 58 drill in a piece of metal, bent as shown and glued to the cardboard of the moving arm. The moving arm is pivoted on a small machine screw at the "eye" end and is adjusted up and down by a wooden knob glued on near the "scale" end.

The distance from sighting hole to cross-wires in this instrument happens to be 37.5 cms, and the distance from sighting hole to scale is 38.0 cms. (This is not at all critical, but the latter must be carefully measured to obtain good results when calculating and marking on the scale.) The "rod" is 8.00 feet long, fitted at each end with a cardboard "target" to improve the precision of sighting. It is illustrated in position PQ of Figure 1. The main quadrant is mounted on tripod or on a stand so that it can be rotated in its own plane and may be clamped firmly in any position.

Results: In use the instrument is set with its sighting hole above one end of the distance to be measured and the rod is placed at the other end, perpendicular to the line of sight.

The main quadrant is adjusted and clamped with its lower cross-wire on the lower target. The upper cross-wire is then brought accurately to the upper target and the distance read from the scale. Typical results are shown in Table 2.

TABLE 2

Scale reading	Actual distance
25.4 ft.	26.0 ft.
31.6	31.2
38.5	39.4
45.0	45.0
59.5	60.0
69.0	70.0

It can be seen that in general the results are correct within 2 per cent. The instrument can be used on non-level ground and to a maximum distance of about 100 feet. Students find it reasonably simple to construct and fascinating to use.

Letters to the editor

Dear Editor:

I was "a-mazed," in reading "An Excursion into Labyrinths" (THE MATHEMATICS TEACHER, May 1958), that the author failed to describe a most simple method of threading a maze such as the one at Hampton Court. The method, which does not require marking the walls of the maze, is simply to walk through while keeping one's right hand in contact with the right wall at all times (or by always keeping in contact with the left wall). This method will always work as long as there are no "free-standing" walls, and so will work for the Hampton Court maze.

While this may not be the most efficient method of threading a maze, it avoids exploring each corridor, and has an added advantage since the maze can be traversed in complete darkness.

Very truly yours,
SAM MATLIN
Chelsea Vocational High School
New York, New York

Dear Editor:

In the hope of being introduced to the "new mathematics," three of our faculty attended an all-day meeting at the University of Kentucky a year ago last spring. Many learned papers were read, and as the day progressed we realized that we had no background for following them. Moreover, most of the vocabulary was new to us. Finding that our notes consisted largely of new terms we intended to look up, we made the following report to the rest of our faculty:

F IS THE REALS ON A FINITE ORTHOGONAL BASE BY 3 NONDEGENERATE QUADRATIC FORMS

To function or not to function,
That is the differential!
Whether 'tis nobler in the mind
    to expand continuously
Or to multiply upon a line to a summation,
And so approach infinity.
A set—a field—and more!
And in those semi-groups
    what isomorphic thoughts may come
Determine flaws.

Ruth Housek
Muriel Hutchinson
Lisle Turner

Respectfully submitted,
MURIEL HUTCHINSON
Margaret Hall High School
Versailles, Kentucky

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

A history-of-mathematics chart

by Julia B. Shelton, Eastern Montana College of Education, Billings, Montana

It is just as absurd to study mathematics without studying its exciting history as it would be to study music without mentioning Beethoven, or Bach, or Copland. If literature were studied without noting author or time or place, it might soon rival mathematics for low place in the popularity poll.

The history of mathematics is the kernel of the history of people and their civilizations. All other histories—of politics, literature, music, art—bear on and relate to the development of mathematics. To keep this history secret is to rob mathematics of much of its glamour and human appeal and to prevent a full grasp of and perspective on the interrelationships of the facets of culture. The lives of the men of mathematics are stories of struggles and victories, of incomprehensible genius, unmatched in any field.

This must be made significant to the student during the high school years if it is to have real and lasting meaning. The big problem is that there is so little time. A large chart hanging in the classroom, showing the names and the major accomplishments of the important mathematicians, and arranged according to fields and in chronological order, could be helpful without being time-absorbing. Day in and day out, such a chart would familiarize the student with the important names and events in the development of mathematics. Perhaps it would motivate the gifted boy or girl to do some research work and arouse interest in ancient number systems and the arithmetic of Babylonia and early Egypt. It should help the young mathematician to form a concept of mathematics as a continuing, changing, dynamic thing.

Students could be encouraged to construct their own charts, selecting specific times, such as the Hellenistic period or the Renaissance. Those students with interests in other fields, such as art, music, ancient or medieval history, might make charts relating mathematics to their particular fields, noting cause and effect and other relationships.

Thus the foundations may be laid for an intensive study of the history of mathematics in college and/or for the study of the history of civilization.

The accompanying reproduction illustrates a history-of-mathematics chart that was constructed as a project while studying "History of Mathematics" at Eastern State College of Education, in Billings, Montana. The master chart has been reproduced on a standard white window shade, with roller mounted on a board bearing eyelets for hanging in the classroom. Smaller photostatic reproductions, folded and punched to fit a standard loose-leaf notebook, were also made from the master chart.

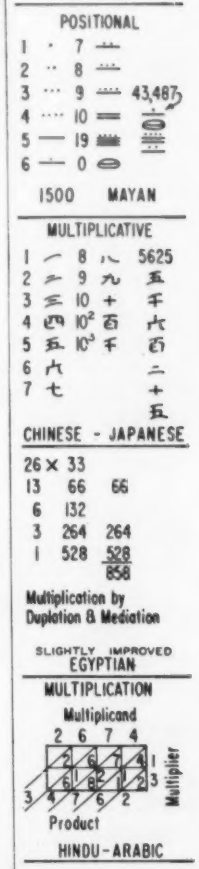
HISTORY OF MATHEMATICS

TIME	ASTRONOMY (AS)	TRIGONOMETRY (T)	GEOMETRY (G)	CALCULUS (C)	ALGEBRA (AL)	ARITHMETIC (AR)	COUNTING & COMPUTING
	−4700 BABYLONIAN CALENDAR −4241 EGYPTIAN CALENDAR		−2900 GREAT PYRAMID OF GIZEH			−2400 BABYLONIAN TABLETS OF UR −2200? NIPPUR TABLETS MAGIC SQUARES	ADDITIVE 1 10 10^2 10^3 1232 −3400 EGYPTIAN
−2100 HAMMURABI OF BABYLONIA		−1750 PLIMPTON 322 BABYLONIA PYTHAGORIAN TRIPLETS			−1850 MOSCOW PAPYRUS EGYPTIAN −1650 RHIND PAPYRUS EGYPTIAN −1600 YALE TABLETS BABYLONIA CUBIC EQUATIONS		
−1600 GRAND AGE IN CRETE	−1500 EGYPTIAN SUNDIAL					−1167 HARRIS PAPYRUS EGYPTIAN −1105 CHOU-PEI CHINESE	ADDITIVE 1 10 Subtractive 38 −2000 to −200 BABYLONIA
−750 ROME FOUNDED			−600 THALES Demonstrative Geometry −540 PYTHAGORAS Postulational thinking (AR) (AL) Music Mystics −500 SULVASUTRAS Hindu (G) & (AS)				CIPHERED 1 alpha 8 eta 2 beta 9 theta 3 gamma 10 iota 4 delta 20 kappa 5 epsilon 30 lambda 6 diagamma 40 mu 7 zeta 100 rho 137 −450 IONIC GREEK
−450 PERICLES			−450 The Three Famous Problems of Antiquity (duplication of cube, trisection of angle, quadrature of circle)	−450 ZENO Paradoxes of motion −430 ANTIPHON Gr. Method of Exhaustion			ADDITIVE 1 5 10 10^2 10^3 10^4 2857 −300 ATTIC GREEK
−431−404 PELOPONNESIAN WARS							
−380 PLATO'S ACADEMY			−370 EUDOXUS Method of Exhaustion −350 MENAECHMUS Conic Sections	−410 DEMOCRITUS Method of Indivisibles Atomic Theory			ADDITIVE 1 5 10 50 10^2 500 10^3 MDCCCXXXIIII 1944 −200 ROMAN
−332 ALEXANDRIA FOUNDED							

	CHUS Copernican system		Elements of Geometry			Perfect Numbers −300+876 Hindu-Arabic Numerals
−60 JULIUS CAESAR		−140HIPPARCHUS (AS) Star Catalogue	−225APOLLONIUS Conic Sections −225 ARCHIMEDES (C) Method of Equilibrium, Infinite Series, Mechanics, Hydrostatics			−230 ERATOSTHENES Prime Number Sieve, Size of Earth
BC						
AD		50? HERON Machines, Surveying			50? DIOPHANTUS Syncopation of Greek Algebra (AR) Number Theory	
100 PLUTARCH		100 MENELAUS Spherical Trigonometry				100 NICOMACHUS Number Theory
	150 PTOLEMY Planetary Theory, StarCatalogue,Trig., Table of Chords		300 PAPPUS Mathematical Collection Centroid Theorems			320 IAMBLICHUS Number Theory
400−1000 DARK AGES			390 THEON Edited Euclid's Elements			
			410 HYPATIA 1st Woman Mathematician			
	505 VARĀHAMIHIRA Hindu Astronomer (T) Table of Sines		470 PROCLUS Valuable Commentary			
622 MOHAMMED					628 BRAHMAGUPTA Cyclic Quadrilaterals & (AS)	710 BEDE Calendar & Finger Reckoning
871 KING ALFRED					820 AL-KHOWÂRIZMÎ	
	920 AL-BATTÂNÎ Moslem Astronomer (T) Law of cosines in spherical trigonometry	980 ABÛ'L-WEFÂ Trig. Tables & Geometry			1000 ALHAZEN Alg., Geom., & Optics	
					1100 OMAR KHAYYAM Cubic Equa. & Calendar	
1215 MAGNA CHARTA					1150 BHĀSKARA Indeterminate Equation	
1271 MARCO POLO			1202 FIBONACCI 'Liber Abaci.' Geom. (AL) & (AR)			
		1470 JOHANN MÜLLER (REGIOMONTANUS) Translator of Math. 'De Triangulis Omnimodis'				
1492 COLUMBUS			1500 LEONARDO DA VINCI Geom. Art, Mechanics		1506 DEL FERRO Cubic Equations	1494 PACIOLI Double-entry Bookkeeping (AL)

TIME	ASTRONOMY (AS)	TRIGONOMETRY (T)	GEOMETRY (G)	CALCULUS (C)	ALGEBRA (AL)	ARITHMETIC (AR)
	1530 COPERNICUS Planetary Theory & Trig.		1510 ALBRECHT DÜRER Projective Geom., Art		1542 ROBERT RECORDE 1545 TARTAGLIA Cubic Equa., Artillery Prob. CARDANO Cubic Equations FERRARI Quartic Equations	
1588 SPANISH ARMADA DEFEATED		1580 VIETA Plane & Sph. Δ's-6. Trig. func. Theory of Equa. (AL)				1553 STIFEL Number Mysticism (AL) 1590 STEVIN Decimal fractions
	1600 GALILEO Falling bodies, Pendulum, Projectiles, Telescopes 1610 KEPLER Laws of planetary motion, Principle of Continuity				1600 HARRIOT Algebra, Symbolism	
1620 PLYMOUTH SETTLED PRINTING PRESS IMPROVED			1635 FERMAT Analytic Geometry (AR) 1637 DESCARTES Analytic Geometry 1640 DESARGUES Projective Geometry	1630 CAVALIERI Method of Indivisibles Logarithms	1630 OUGHTRED Algebra, Symbolism, Slide Rule, 1st table of natural logs	1614 NAPIER Trig., Logarithms, Computing rods 1624 BRIGGS Logs to base 10 1635 FERMAT Theory of Numbers Probability
		1650 N. MERCATOR Mercators Series, Astr., Cosmography	1650 B. PASCAL Conics, Cycloids, Probability, Computing, Pascal triangle, Religion	1650 WALLIS Precalculus, Imaginary numbers		
	1670 CHRISTOPHER WREN Astronomy & Architecture		1672 MOHR Consts. with compasses only	1670 J. GREGORY Infinite Series BARROW Differentiation 1680 ISAAC NEWTON Fluxions, Gravitation, Hydrostatics, Dynamics, Experimentalist, Principia et al.	1670 HUYGENS Probability	
	1690 HALLEY Astr. & Life Ins. Tables			1682 LEIBNIZ Symboliclogic, Calc., Determinants, Computing Machines 1720 TAYLOR Infinite Series	1690 J. BERNOULLI Probability 1720 DE MOIVRE Probability, Actuarial Math	

COUNTING & COMPUTING



1733 SACCHERI
Non-Euclidean geometry prelude

1740 MACLAURIN
Fluxions, Curves

1750 EULER
Writer of Mathematics, Notations & Formulas

1770 LAMBERT
(π irrational, hyperbolic functions, map projections)

1776 UNITED STATES INDEPENDENCE

1780 LAGRANGE
Calculus, Mechanics. Theory of Numbers

1789 FRENCH REVOLUTION

1794 MONGE
Descriptive Geometry

1805 LEGENDRE
Elements of Geometry, Number Theory (AR)

1805 LAPLACE
Differential Equations, Celestial Mechanics (AS), Probability

1806 BRIANCHON
Duality

1806 ARGAND
Imaginary numbers

1807 ROBERT FULTON'S STEAMBOAT

1820 GAUSS
Non-Euclidean Geom.
Theory of numbers (AR)
Fundamental Theorem of Alg.

1820 FOURIER
Series expansions

1824 ABEL
Alg. sol. of gen. quintic impossible elliptic funcs.

1825 J. BOLYAI
Non-Euclidean Geom.

LOBACHEVSKY
Non-Euclidean Geom.

1830 PONCELET
Projective Geometry

1830 CAUCHY
Rigorization of Analysis. Definitions

1830 GALOIS
Groups, Theo. of Equations

1840 STEINER
Modern Geometry

1850 CAYLEY
Matrices, Determinants, Hyperspace, Invariant theory

1850 LIOUVILLE
Transcendental numbers

1853 HAMILTON
Quaternions

1854 GRASSMANN
Ausdehnungslehre

1854 RIEMANN
Riemannian Geometry
Riemann Surfaces

1875 BROCARD
Modern geometry of triangle

1881 GIBBS
Vector analysis

1880 GEORG CANTOR
Irrational numbers
Transcendental numbers
Transfinite numbers

1888 LEMOINE
Geometrography

1890 WEIERSTRASS
Arithmetization of Analysis

1891 HEAVISIDE
Vector analysis

1905 EINSTEIN
Theory of Relativity, Atom bomb

NOTES

Dates given denote approximate time work was done.
Minus date is B.C.
Division lines between departments of mathematics are arbitrary.

This information taken from "An Introduction to the History of Mathematics" by Howard Eves, March 1959.

Benefits derived from mathematics projects

by Dorothy Fish, Pandora-Gilboa High School, Pandora, Ohio

Students in our high school, enrolled in any elective mathematics courses, are encouraged to undertake mathematics projects, which are then entered each spring in the annual District Science Day Contest sponsored by the Ohio Academy of Science. Winners in the District Contest are eligible to compete for awards in the State Science Day Contest. This contest creates great interest and enthusiasm in our school, and to win a "superior" rating in the State Contest is considered one of the highest honors that a student can achieve.

From a scholastic viewpoint, the purposes of a mathematics project are twofold: (1) to provide a challenge to the bright student to delve into some particular topic far beyond the requirements of the regular curriculum, to the full extent of his ability; and (2) to give a student a glimpse of the fascinating study of the history of mathematics, so that the names of Euclid and Pythagoras and other great heroes of the past will mean living men whose devotion and efforts in furthering the development of mathematics have led to the many complex and varied branches of modern mathematics so essential in our world today.

Projects may be in the form of oral demonstrations, models, or research papers. I believe that projects should be on a voluntary basis, since only those students who have the time and inclination to spend hours on outside work and study will be able to produce creditable results. Interest in projects is readily created by throwing out a casual remark, an anecdote, or a reference to something of historical value in connection with the current assignment. The keener students are learning to seize these suggestions with alacrity, recognizing them as ideas for possible projects.

When introducing the unit on fractions in a freshman algebra class last fall, I spent possibly five minutes giving illustrations of the methods used by the early Babylonians and Greeks for writing fractions. I explained the Egyptian concept of fractions as a part of the whole, and questioned the class as to what a fraction meant to them. After the lesson, a bright little girl came up to ask me whether she might write a paper on the history of fractions. In her study, she classified fractions under the four main headings: unit, sexagesimal, common, and decimal. Then she traced the various concepts of fractions held through the ages down to the modern view of fractions as possessing many of the properties of whole numbers. This girl has acquired, as a result of her project, a glimpse of the slow process of the historical development of mathematics.

In a class in plane geometry, even the names of some of the important propositions—"The Bridge of Assess," "The Pythagorean Theorem," etc.—suggest a rich field of topics for suitable mathematics projects. A sophomore, with training in industrial arts, made wooden models showing the dissection proofs of the latter theorem, as given by the early Chinese and later by the Hindu Bhaskara. He then collected about fifty other proofs of this famous theorem, including the one attributed to President Garfield. The inclusion of a short biography of Pythagoras and his influence on later mathematics makes this an interesting and worthwhile project.

Advanced mathematics courses offer a

great variety of bypaths and excursions into fields related to calculus, logarithms, special curves, etc. A unit on the solution of cubic equations last fall resulted in a "superior" project on the life of Omar Khayyam. A study of this Persian mathematician's contributions to various fields of mathematics, including a reproduction of his geometric method of solution of a cubic equation, produced a research paper as good as almost any paper written by an advanced mathematics student in college.

A chance reference to the cycloid curve, as an illustration of the use of parametric equations, interested a senior in delving into the history and properties of this remarkable curve. He made models of the cycloid curve, the cardioid, the hypocycloid and epicycloid, and he derived the equations of these curves. He learned about the parts Galileo, Leibnitz, and Pascal played in studying this curve, and was fascinated with Bernoulli's proof that it is the brachistochrone curve. He told me later that whenever he saw a pendulum, he would always be reminded of his project on the cycloid.

Students who have produced successful projects often tell me, when they come back to visit a few years later, that the study and research which went into their projects has been of inestimable benefit, not only because of mathematical knowledge and skills acquired, but because a door, hitherto closed, was opened, and they were able to catch a glimpse of the fascinating history and development of that wonderful subject, mathematics.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH MATHEMATICAL ASSOCIATION OF AMERICA

January 30, 1960
Conrad Hilton Hotel, Chicago, Illinois
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Statler-Hilton Hotel, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

JOINT MEETING WITH NEA

June 29, 1960
Los Angeles, California
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

TWENTIETH SUMMER MEETING

August 21–24, 1960
University of Utah, Salt Lake City, Utah
Eva A. Crangle, Board of Education, Salt Lake City 11, Utah

Other professional dates

Mathematics Section, New York Society for the Experimental Study of Education
December 4, 1959
Washington Irving High School, East Sixteenth Street and Irving Place, New York, New York
John A. Schumaker, Secretary, Montclair State College, Upper Montclair, New Jersey

Women's Mathematics Club of Chicago and Vicinity
December 5, 1959
Stouffer's Restaurant, Randolph and Michigan, Chicago, Illinois
Sarane Starr, University of Chicago High School, Blaine Hall, Chicago 37, Illinois

• NEW IDEAS FOR THE CLASSROOM

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota.

Decimals and duodecimals, .333 . . . or .400

by Robert C. McLean, Jr., Los Angeles, California

When the theory of numbers professor enters upon the discussion of never-ending decimals, he carefully points out that irrational numbers give non-repeating digits and that common fractions give repeating digits. A common example of this latter condition is one-third, well known as .33333. . . .

This bit of insight is probably as much as most of us high school mathematics teachers carry away from the course, and we use it at odd moments to add to the scope of our own courses. There is more to the story than this. Although we are probably aware of it in a hazy way, we usually do not go on to explain to our students that common fractions may be expressed by ending decimals by the right choice of base for the numerical system of notation used. Irrational numbers, on the other hand, are always unending decimals, no matter what the notation.

The title of this little essay includes one-third written to the base 10 (decimal) and one-third written to the base 12 (duodecimal). Because of our extreme familiarity with this number in the one system of notation, we calmly go through life treating one-third as if it always had an unending notation, little realizing that by changing bases, we could achieve a simple equivalent for it.

Now that you have thought about it, it is obvious that any factor of twelve will have a one-digit duodecimal equivalent for all common fractions in which it is a denominator. For example, one-half is .6 and one-sixth is .2, one-third is .4 and one-fourth is .3.

Furthermore, any denominator which is a factor of 144 will give a two-figure duodecimal equivalent. Let us try a few. One-eighth is .18, and one-eighteenth is .08. One-ninth is .16 and one-sixteenth is .09. Simple, is it not?

Just as we can express any common fraction with a denominator which contains only multiples of powers of 2 and 5 as ending decimal fractions, so can we express any common fractions with a denominator which contains only multiples of powers of 2 and 3 as ending duodecimal fractions.

A simple outline of the reasoning which would explain this phenomenon would take these things into consideration. First, the prime factors of 12 are 2 and 3. Twelve is expressible as $2^2 \times 3$. As all the denominators of duodecimal fractions are powers of twelve, any possible denominator is expressible in the form $(2^2 \times 3)^n$, where n is an integer. Any common fraction is expressible as an exact duodecimal fraction only if it can be expressed with one of these powers of twelve as a denominator.

Symbolically, if a/b is a fraction with a and b mutually prime and b not 0, then, by proper choice of an integral multiplier k, we can alter the form of a/b to the equivalent form ka/kb such that kb equals

$(2^2 \times 3)^n$. Thus the only possible factors of kb are powers of 2 and 3. So, b must have as prime factors only 2 and/or 3.

For the example above, where we expressed $\frac{1}{8}$ in duodecimal notation, k is 18 and b is 8. For the fraction $\frac{1}{18}$, b is 8 and k is 18.

Another interesting system in which to investigate examples is the binary sytem. Here, all fractions with denominators which are not powers of two are expressible only as unending decimals. Some mathematicians support the use of some prime number such as 7 or 11 as the base for numerical notation because in these notations, if the fraction is expressible exactly in place-value notation, then it is a fraction whose denominator is a power of the base.

It is interesting that one-half expressed to the base 7 is .333333. . . . Consider one-half expressed as a common fraction with denominator 7. In decimal numbers, this is $3\frac{1}{2}/7$. That 3 of $3\frac{1}{2}$ is the first 3 after the point of .333333. . . . Consider the remaining $\frac{1}{2}/7$; changing this to 49ths (the second power of seven and the place value of the second place after the point) we find we have $\frac{1}{2}/7$ equals $3\frac{1}{2}/49$. (Just multiply the numerator and the denominator of $\frac{1}{2}/7$ by 7 and see for yourself.) Putting this 3 after the first 3 and going to the next denominator, we have $\frac{1}{2}/49$ to change to 343rds (343 is 7^3). This gives us $3\frac{1}{2}/343$. There is another 3 for the series, and another $\frac{1}{2}/7^n$ to be dealt with. If you wish to pursue the matter further, you may. Needless to say, all you will get is another $3\frac{1}{2}$ over some power of seven.

This is where we came in. We found that .33333 . . . (decimal) was really .40000 (duodecimal), and that .500000 (decimal) was really .33333 . . . (septuagesimal). As the man said, figures can't lie!

Now, lets see what notation to use to illustrate these results. After all, you just cannot write that $.3333_{10}$ is equivalent to $.4000_{10}$, because 10 looks very much like 10. You have to know that the first one-zero was ten and the second one-zero was twelve—in their respective systems. People who go around writing these as $.3333_{10}$ and $.40000_{12}$ are just plain inconsistent. In the first expression, both numbers are decimal numbers; in the second expression, one number is purported to be duodecimal and the other is decimal! As if *ten* were the *only* base for respectable notation! Probably $.40000_{\text{twelve}}$ is the best notation though some have suggested :40000.

Solutions to Problems in "The Mathematics Student Journal," November 1959

Problem 140. If Wat won, Wat's statement, "George won. I came in last," contains two lies. This is ruled out, so Wat did not win. Adolf's second statement must therefore be true. Adolf was second. So Billy was fourth (Billy's statement). Wat must have been third (Tommy's statement); George was first (Wat's statement); and Tommy was fifth (George's statement; also, the only place left).

Problem 133* (March 1959) gave a problem of this type and challenged students *to compose an original problem of this nature.* No student sent in an original problem. Can you stimulate your students to compose one now?

Problem 141. Let S be satellite, SAB the diameter of the Earth that passes through S, ST a tangent from S to the Earth. By a well-known circle property, $ST^2 = SA \cdot SB$. But $SA = 100$, $SB = 8100$. Hence, $ST = 900$. Let O be the center of the Earth, and $\theta = \angle SOT$. Since $OS = 4100$, $OT = 4000$, $\cos \theta = 40/41$. Area generated by rotating arc AT about diameter AB is given by $2\pi r^2(1 - \cos \theta)$ as being about 2,450,000 square miles. This is rather more than 80 per cent of 3,000,000 square miles.

Problem 145. 457 is being multiplied by 38.

A column of unofficial comment

Arithmetic and teacher preparation

by Francis J. Mueller, University of Rhode Island, Kingston, Rhode Island

To take issue with a Report of the College Entrance Examination Board's Commission on Mathematics,* in view of the quality and magnitude of that body's contribution to mathematics education, may seem to border on the profane. Yet, because of this quality and magnitude, and the consequent general deference that many will *automatically* pay to its recommendations, I feel obliged to point out one important implication that should be looked upon with an extremely wary eye.

I have reference to Chapter 5, pages 48–50, where the Report comments on "the education of elementary school teachers."

People truly interested in the improved preparation of elementary teachers of arithmetic will find nothing to quarrel with and much to commend when they read, " . . . one cannot teach any subject well unless his knowledge exceeds the material he is expected to teach," " . . . knowledge of teaching methods and of educational psychology is no substitute for a sound understanding of . . . basic mathematical concepts," and the quotation from Claude V. Courter, which ends, "Only when pupils understand the reasoning behind simple mathematical ideas are they ready to comprehend more difficult concepts."

When, after citing these arguments, the Report goes on to say that "at least three years of secondary mathematics should be required" of prospective elementary teachers, we who are concerned with the preparation of those teachers must applaud. But later in that very same paragraph an uneasiness sets in as one reads, "A substantial contribution to the improvement of mathematical instruction in America would be made by providing elementary school teachers with a background in mathematics *adequate to assure their competence in teaching arithmetic.*" (Italics mine.)

Does the Report mean to imply that all or even most of the secrets of arithmetic understanding can be found in the study of algebra, geometry, and trigonometry? Apparently so, for the Report goes on to suggest that all students entering teachers colleges have at least the first three years of high school mathematics, as recommended by the Commission. Moreover, facing the reality that this is not likely to be accomplished in the near future, the Report implies that until it is accomplished, this work should be offered in teachers colleges. And for those teachers who have their college education behind them, the material of these three high school years should be the substance of in-service programs.

I wish to repeat emphatically that the more mathematical background a teacher of arithmetic possesses the better; but to suggest that the ideal content preparation

* "Program for College Preparatory Mathematics," *Report of the Commission on Mathematics* (New York: College Entrance Examination Board, 1959).

for the teacher of arithmetic lies in the study of algebra, geometry, and trigonometry is as questionable as to suggest that the ideal preparation for a good tennis player lies in practicing Ping-pong and badminton.

Background is fine, but let us not mistake the wrapping for the package. If our intention is to turn out a good and knowledgeable teacher of arithmetic content, then the primary object of study for that teacher should be the subject of arithmetic. There are far too many administrators and general education specialists now who contend, in effect, that if the prospective teacher knew enough *arithmetic* to pass out of the eighth grade, then she knows enough *arithmetic* to go back and teach that subject at any grade level below the ninth. Unfortunately, the Commission's report can very easily be construed to bolster that position.

Nowhere in the mathematical topics suggested by the Report for grades nine, ten, and eleven—the three years of high school mathematics it recommends—can elementary teachers find answers *that they can use* to meet the everyday questions, "Why do we carry?" "How can we subtract a 6 in the ones' place from a 3?" "Why do we invert the divisor?" "Why do we indent?" "Why does the decimal point in the answer go there?" and countless other everyday "why's" that are similar in character.

Granted, subsequent phases of mathematics throw welcome light on areas and techniques that are commonplace in arithmetic; but the crucial point is that what most concerns the teacher of arithmetic content is *assumed* in the study of algebra.

Few people with relatively extensive college mathematics backgrounds (by current standards, say, through differential equations) find their way into teaching at the elementary level. In the very few instances that I know personally, these mathematically better-prepared people were not nearly as effective at teaching arithmetic as were colleagues whose teachers college mathematics amounted to a single three-hour course in the content of arithmetic. Nor can this disparity be attributed to the lack of knowledge of arithmetic methods on the part of the former: it was clearly the lack of specific *arithmetic knowledge* that troubled them.

It is a well-known fact that the zeal of the convert is often more intensive than that of one born to the faith, and perhaps this is true in my case. When I accepted a position on the mathematics staff of a teachers college some eight years ago, fresh from teaching the usual mathematics subjects in a liberal arts college, I was amused when I was assigned a course in the content of arithmetic, and remember commenting, "How can I teach for a semester on this—what everybody knows?" But I have yet to crowd into the allotted semester all I have wanted to, and I'm constantly learning more. The mature and serious study of the content of arithmetic is not only worthy of a place on the college program of a student preparing to enter elementary teaching, but, I contend, it represents the most valuable single investment of that student's mathematics-course-taking time.

"Traditional mathematics—algebra, geometry, trigonometry—is still the great core of the subject, but the distance between the core and the newer developments must be shortened by new methods of approach."—"Program for college preparatory mathematics," *Report of the Commission on Mathematics, College Entrance Examination Board, New York, 1959, p. 6.*

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Algebra Accelerated, Book I, E. Justin Hills and Estelle Maziotta (Peoria, Illinois: Chas. A. Bennett Co., Inc., 1959). Cloth, 320 pp.

A casual look at the colorful exterior of this textbook, the unusual chapter titles, the use of color throughout, and the "space age" pictures leads one to expect a treatment of content equally up to date.

However, an examination of the content reveals that the worth of the book as a high school text must be challenged from two points of view.

First, it does not meet the requirements suggested by a number of the groups presently recommending curricular revision, e.g., the Commission on Mathematics. In their publication *Program for College Preparatory Mathematics* we find: "The new emphasis in the study of algebra is upon the *understanding* of the *fundamental ideas* and *concepts* of the subject, such as the *nature* of the *number systems* and the *basic laws* for addition and multiplication (commutative, associative, distributive). The application of these laws in various number systems, with emphasis on the generality of the laws, the meanings of conditional equations and identities and inequalities, is stressed." Taking this small portion of the suggestion, let us examine the book in question. Will it further the student's understanding of the number systems? On page 40, the authors depict the "algebraic number scale" and say "to the right are positive signed numbers with which you are familiar. They are used to count units or things, etc. To the left are the negative signed numbers which, so far, are strange to you." As you see, counting numbers are *not* distinguished from positive signed numbers, so that, from the beginning, students are hindered in their understanding of the nature of the number systems.

Then, too, the basic laws for addition and multiplication are not mentioned in this text even though their understanding is so necessary to a complete understanding of operations on algebraic expressions. The commission stresses emphasis on conditional equations *and* identities *and* inequalities. According to the *Instructor's Key* for this text, "This important topic [inequalities] is introduced to counter the stress on equations." (Whatever that means.) Work on inequalities is not woven into the structure of the text, and no inequalities are introduced in the tests and reviews found at the close of each chapter. It is evident the authors do not consider the treatment of inequalities important. This analysis of the text could be carried further but readers will agree that, with these *basic* weaknesses in this book, there is no foundation on which to build the kind of understanding suggested by the Commission report.

The second point of view from which the worth of this book should be challenged is that it does not fulfill the claims of its authors. In the preface to the teachers, the authors suggest, "You will find the brevity, conciseness and precision of the book consistent with the ideals of mathematical thinking." A careful search of the book reveals no challenge to the student to think but only instructions to follow a set of *rules*. That philosophy of the text is revealed on the first page of the first chapter where the authors say, "Master these simple rules and you can become a winner" and, "You will learn a set of rules dealing with algebraic numbers." In each chapter thereafter rules are stated following a few examples solved by the authors. Since examples and rule occur often on the same page, there is little likelihood that understanding has been developed.

Another claim the authors make is that of precision. This quality, if it exists, should be found in rules, definitions, and the like. Among the many instances of the lack of precision in the text, consider these: "The sum of two numbers having opposite signs is their arithmetic difference (their difference without regard for sign) and has the sign of the larger number" (p. 41). "Notice that a square of a minus signed number is the equivalent of two minus signed numbers, while a cubed minus number implies three minus signed numbers" (p. 52).

Among the definitions in the glossary is "Equivalent equation: an equation all of whose coefficients and constant are exact multiples of the coefficients of another equation." Is the equation $7x+5y=30$ then equivalent to the equation $x+y=5$?

The omission of certain of the so-called unusual features of the book (among them, the samples of carelessly done students' work and the poorly constructed completion tests at the close of each chapter) might improve the text but could never make it acceptable.—*May Blair, Pekin, Illinois.*

Arithmetic in My World, 9, C. Newton Stokes, Anne Beattie, and Ruth I. Hoffman (Boston: Allyn and Bason, Inc., 1959). Cloth, 308 pp., $3.60.

In this review I am assuming that the book is to be used in classes where the students are not capable of taking a course in algebra.

The authors have presented arithmetic in a framework of social problems. In the first four chapters the operations of addition, subtraction, multiplication, and division are studied under "Exploring Your Interests," "Your Needs for Money," "Out-of-School Problems Experiences," and "Part-Time Work." The arithmetic operations with fractions, all three cases of per cent, decimal fractions, and some compound interest problems are also presented. Good review exercises are found throughout the book. There are a large number of word problems and a small amount of drill exercises. The authors have tried to design the word problems for a more mature student, although this is difficult to do considering how simple some of the problems must be. The last half of the book concerns "Mathematics in Family Affairs," which deals with money transactions, income tax, social security, insurance, food buying, installment buying, and travel, without introducing new concepts of mathematics. These subjects might overlap too much with social studies courses, home economics, or health courses to be of interest to many of the students. Also, there might be some debate as to whether the mathematics teacher should teach these subjects. In the early chapters the authors emphasize that the student should make a quick approximation of numerical answers. The authors do present some interpretation of graphs, but I feel that there could be more in this type of book.

The book uses color to emphasize points. The type is large and well-spaced for reading. However, I feel that there is too much printed material. A person with a reading problem would be seriously handicapped. The explanations are lost in too much type for each page. The student who will be reading this material probably will have too short an attention span to derive much value from many of the written explanations.

Mathematically speaking, I feel that many of the explanations are confusing and that within some of the units the order of presentation could be improved. The authors define subtraction and division in terms of addition and multiplication, respectively. They do not discuss the meaning of a number before beginning the discussion of the operations. Here is where some of the work being written by groups such as the Ball State Group or the School Mathematics Study Group would be valuable to the weak student.

Algebra is not introduced until the last chapter and is inadequate. An earlier introduction of simple algebra would have provided a break in the routine of the book, would have given the student a feeling of something different, and would have made many of the authors' explanations clearer (i.e., in per cent they use $p=br$, $r=p/b$, and $b=p/r$). The symbol "¾" as an alternate for "3÷4" is not made clear, thus making necessary such operations as $\frac{3}{4}=7\frac{1}{2}/10=75/100=.75$.

The authors present several methods of performing the same operation, and introduce terminology with each method, but I feel they do not follow all of them to the point where the student has complete understanding. This, with the typography, presents a confusing picture.

This book is essentially an eighth-grade text with some more complicated or mature topics added. If the students have not had much consumer mathematics in the seventh and eighth grades, and if one agrees with the authors that the subject matter is of interest to ninth-grade students, then a school may be interested in the book. I feel that its weak points are: it is too wordy; some explanations are confusing; the repetition of subject matter may tire the student. In connection with the last point, more interruptions to investigate simple algebra and geometry could be included. This book's strong point is the application of arithmetic to everyday problems.—*Donn L. Klingler, Lyons Township High School, La Grange, Illinois.*

The Gentle Art of Mathematics, Dan Pedoe, New York, The Macmillan Company, 1959. Cloth, 143 pp., $3.50.

This thin book, in the traditional British pattern of mathematics for fun and enjoyment, continues a long succession of such books. The plan is largely one of collecting into a chapter well-known related puzzles which can be explained by developing a minimum of common theory. The reading is enlivened by the historical comments and the digressions. These range from Eliza Doolittle to two letters on probability from Isaac Newton to Samuel Pepys.

Among the well-known topics considered are the binary and ternary scale, the game of Nim, the twelve-coin problem, the St. Petersburg Paradox, Buffon's Needle, infinite classes, the escape artist's trick, supplying three houses with main services, the Moebius band, the four-color problem, and the class of all classes.

Although the author states that "the book is intended for the many people who would like to know what mathematics is about, especially modern mathematics," the spirit of the book and the topics chosen are essentially nineteenth century, with only slight excursions into the twentieth century.

The book should be useful in introducing a bright student to a variety of mathematical concepts, even though the theoretical development may be too elliptical.—*Rothwell Stephens, Knox College, Galesburg, Illinois.*

Intermediate Algebra, Lyman E. Kells (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1959). Cloth, xi+296 pp., $4.95.

This text covers the traditional material of the one-semester intermediate algebra course, for the greater part in the conventional manner. The content and terminology vary so little from the traditional course that this reviewer had the impression that this text was a revised version of a previous edition. This is not a revised edition.

Intermediate Algebra is intended for college students with little or no background in algebra. Although the traditional material is presented on a mature level, it would seem to this reviewer that the rigor of this development prevents the student with little background in algebra from gaining much insight into the spirit of mathematics. The polished version of mathematics presented in this text leaves little room for student explorations. The sequence of definitions, rules, examples, and related exercises is the pattern used to develop each new topic.

Inadequacies found in so many high school texts today appear in this college text. Several instances will support this statement. On pages 1 and 2, the following appears: "This chapter deals for the most part with the so-called *natural numbers*: zero and the positive integers of arithmetic." Reference to "positive integers of arithmetic" only confuses the distinction the mathematician would make between the set of natural numbers and the set of real numbers. On page 35 is found the rule for subtraction of real numbers. "To subtract a second number from a first, change the sign of the second and add." Clearly no distinction is made between a number and the symbol used to refer to that number. Further, word problems are classified according to types of problems and the factoring of trinomials is also developed by types. The previous instances of the content of this text merely indicate, in the opinion of this reviewer, the vintage of this text.

Some emphasis is given to the structure of mathematics with particular emphasis on the basic distributive law. This is good. However, the commutative and associative laws should be given equal emphasis if the structure of algebra is to be a part of the pedagogical development of a text. What can be gained by stating the associative law as it appears on page 6? "The sum or the product of several numbers is found by the usual processes of arithmetic; in this process . . . the numbers may be partitioned in any way into non-overlapping parts for obtaining partial sums or products from which to derive the final results." If the use of numerical variables is used to impart the meaning of the distributive law, certainly this justifies the use of the same means for clarifying the commutative and the associative laws. A footnote on the same page does this but this clarity is lost in the main body of the text.

The format for this text is well done; the type is of good size; the paper is of excellent quality. The reviewer was confused by the meaningless expressions which appear on the front cover, e.g.,

$$\text{“} \frac{4b}{x+3)(x} \text{.”}$$

If they appear merely to attract attention, then they accomplish this purpose.

There are quite extensive lists of exercises, mostly of a routine nature intended for drill. Of particular interest are the special "topics for discussion" which appear throughout each chapter. These topics raise thought-provoking questions. Each chapter closes with a list of review exercises. The book closes with a list of answers to the odd-numbered exercises and the usual four-place tables of common logarithms. Supplementary tables of proportional parts appear on the same pages. A four-decimal place table of powers, roots, and reciprocals is also included.

Whether or not *Intermediate Algebra* is selected as a textbook will depend upon the teacher's own theory of teaching algebra. If the teacher feels that the traditional terminology, symbols, and content are sufficient to impart the clearest understanding of mathematics concepts, he will find this text offers a well-organized course of study. If, however, the teacher believes that much can be gained by including the language and symbols of logic and sets, he will find this text unsatisfactory.—*Eugene R. Epperson, Talawanda High School, Oxford, Ohio.*

Intermediate Algebra, Ross H. Bardell and Abraham Spitzbart (Reading, Massachusetts: Addison-Wesley Publishing Company, Inc. 1959). Cloth, ix+274 pp., $4.75.

This book, a college text for the student who has had only two years of high school mathematics, is traditional in its treatment of topics.

Some of the outstanding features which distinguish this book from similar ones are the exclusive use of integers and integral replacements for variables for the first sixteen pages while the student acquires certain basic skills, and the careful treatment given in the chapter on exponents and radicals and in the chapter on exponential and logarithmic functions. In this latter chapter the authors correctly indicate that any positive number except 1 can be a base for logarithms of positive numbers—a point not always made by writers of such books. Here a study is made of the graphs of loci of logarithmic functions for bases greater than 1, but unfortunately the authors fail to complete the study by considering positive bases less than 1.

Two other strong points are the sample exercises throughout the book, and the authors' comment that of the two expressions "$\sqrt{3}/3$"

and "$1/\sqrt{5}$" the determination of which is simpler is governed by the use to which they are to be put.

As this reviewer studied the book this careful treatment came as a surprise because of the almost complete lack of attention to certain details in the first half of the book. Here are some of the gross errors and inconsistencies that one encounters in this book. One of the rules given for operating with real numbers is: "When two numbers of opposite sign are to be added, subtract the number with the larger numerical value, and attach the sign of the number with the larger numerical value." This tells us to subtract one real number from another "of opposite sign." Subtraction of real numbers has not yet been defined. What the authors probably mean to say, if one accepts their definition of "numerical value" or "absolute value," is: ". . . consider their numerical values and subtract the smaller numerical value from the larger numerical value. . . ." (Reviewer's note: What happens if they have equal numerical values? The book gives no indication.) The student's only chance is to look at the sample problems and then come up with his own (usually unverbalized) rule; indeed, even if the given rule *were* correct this is probably what most students would do anyway. Here also it is stated that "the numerical value of 3, as well as of -3, is taken to be 3." Then several chapters later are the directions, "Determine a simple numerical value of the expression in each of the exercises." One of the exercises is "$(-27)^{-1/3}$" for which the answer is given as a *negative number*, $-\frac{1}{3}$.

In Chapter 1, after stating that zero is an integer, another rule is given: "The product of any number of integers is positive if an even number of integers is positive, and is negative if an odd number of integers are negative." This would lead one to believe that zero is sometimes positive and sometimes negative (except that nothing is said in the book about the product of zero and a negative number). Many pages later the authors do correctly imply that zero is distinct from the set of positive numbers and the set of negative numbers, but this utter lack of concern about zero in the first half of the book is devastating. In describing what comprises the real number system, it is not clearly stated that zero belongs to the set of real numbers, nor, in discussing the factor of proportionality for direct and inverse variation, is zero excluded as a possible value.

The commutative, associative, and distributive principles for positive integers, then negative integers, rational numbers, and real numbers, are not mentioned after Chapter 1. How beneficial it would be to see the distributive principle used as the basis for factoring algebraic expressions and manipulating fractions as is recommended by the various mathematics commissions, committees, and programs in existence to improve secondary teaching! The student studies the commutative laws for adding and multiplying numbers. Why not reinforce his understanding by means of questions regarding commutativity with respect to operations of subtraction and division? Similarly, the authors twice mention "closure" for the rational numbers and then for the real numbers. Why not ask questions about closure for the even integers and odd integers and for the complex numbers? All questions in the book don't *have* to be of a manipulative nature to teach the student his "bag of tricks."

In line with recent trends, the section on determinants could be omitted, and algebraic and geometric studies of inequalities could very profitably be included.

The phrase "if and only if" is correctly used by the authors several times, but there is no explanation of its meaning. More than likely the student thinks it is redundant or is "put in for emphasis." Wouldn't a sample be worthwhile to show that it is a short way of indicating the conjunction of a conditional statement and its converse (a bi-conditional)?

After the authors derive the quadratic formula by completing the square they have the student solve as many as 46 equations by completing the square before the student is allowed to use the formula. The cart is before the horse.

In the section on linear equations one is to consider the equation $ax+by=c$, which is shown to graph as a straight line if $a\neq 0$ or $b\neq 0$. The book then states: "If $a=b=0$, then $ax+by=c$ reduces to $c=0$, which is trivial." It may be *trivial* but it is *interesting* because if $a=b=c=0$, then the graph of the locus of $ax+by=c$ is the entire number plane. But the next sentence then says: "Hence the graph of $ax+by=c$ is always a straight line." Contradictions and inconsistencies such as these should be of real concern to the student and the teacher of mathematics at any level of education.—*Howard Marston, The Principia, St. Louis, Missouri.*

Mathematical Models, H. Martyn Cundy and A. P. Rollett (London, The Clarendon Press, 1956). Cloth, 240 pp., $5.50 from Oxford University Press, New York.

It is well known to all of us that a good model or other schematic physical device can provide a real lift in getting across a difficult or new idea. It is becoming increasingly apparent, as more research is done on the subject, that not only is the visual-intuitive approach to much of mathematics the one most likely to succeed with beginners, but also that it is the point of view from which many of the great mathematicians have been led to new discoveries. The book by Cundy and Rollett is written for the secondary-school student and his teacher. It contains a rich store of all kinds of diagrams, devices, and methods for presenting plane and solid geometry, plus much more that is not normally presented by mechanical and physical means.

The plane geometry section begins with some of the beautifully obvious, but generally neglected, proofs of the Pythagorean theorem, and moves from there through loci to a host of fascinating applications. The solid geometry section contains directions for making actual cardboard and Plexiglas three-dimensional models; but the authors do not stop there. They take every opportunity to apply visual and intuitive ideas to theorems and, sometimes, proofs.

The fact that *some* proofs are given makes one wonder why so many were omitted. The reader is often tantalized by the exhibition of an unexpected property of a curve or a surface with no hint as to its derivation. Even if proofs had to be omitted, one would have welcomed more specific references to sources, which are too sparsely scattered throughout the book. There is a rather short general bibliography.

But these faults are minor and are surely outweighed by the general excellence of the presentation. One could almost say that this book does for elementary geometry what Hilbert and Cohen-Vossen's *Anschauliche Geometrie* does for higher geometry; than that there could be no higher praise.—*C. Stanley Ogilvy, Hamilton College, Clinton, New York.*

Plane Geometry and Supplements, Walter W Hart, Veryl Schult, and Henry Swain (Boston: D. C. Heath and Company, 1959). Cloth, ix+454 pp., $3.80.

Many suggestions are now being made concerning the nature of the high school geometry course. Textbook writers need to give consideration to these suggestions. The latest list of such suggestions to be published is that of the Commission on Mathematics. Since these recommendations will probably be as far-reaching as any, they will be used as a basis for reviewing this text.

The Commission lists three objectives for geometry. The first deals with the acquisition of information about geometric figures in the plane and in space. This text includes all the plane and solid geometry content outlined by the Commission, plus a considerable amount more. However, the writers did cut down on the content of plane geometry to allow space for solid geometry, coordinate geometry, a rather extensive treatment of trigonometry, a chapter on reasoning, and other units.

The second objective is "the development of an understanding of the deductive way of thinking and a reasonable skill in applying this method to mathematics situations." This book contains more material of this type than most texts. Much of this material is found in a late chapter. The necessity for having undefined terms and assumptions in a deductive system is mentioned several times, but the discussion in the text does not present a convincing argument for deduction itself. Deduction would avoid the errors of optical illusions, that is true, but so would induction based on accurate measurement. Also it is mentioned that deduction is a means of arriving at true conclusions—the implication being that induction has little to offer in the attainment of true conclusions. It seems that much is left unsaid here—much that would help students sense the role of deduction and induction in the search for truth. The postulates and definitions are distributed throughout the text near the time when they will be needed. Still, most of the definitions are made before students sense the need for them. Some attempt is made to use modern terminology in the exposition. The authors proceed as far in the direction of rigor as they dare and yet do not overshoot the level of understanding of a large percentage of the class. Exercises are included that demonstrate the role of deduction in nonmathematical situations. The Commission recommends several short sequences of theorems to be studied. This text, like nearly all current texts, presents essentially a single long sequence. The authors should be commended for devoting as much space as they have to the discussion of reasoning. However, unless a teacher uses the text materials after careful deliberation and judicious supplementing, most students probably will still not gain much of an understanding of the nature of proof and the role of deduction in the study of mathematics and its applications.

The third objective is "the provision for opportunities for original and creative thinking by students." Such opportunities are provided. True, it would have been desirable if there were more, but what has been done is decidedly more than is found in most texts. A page or two before certain theorems are treated formally and before certain postulates are stated, exercises are provided in which students can formulate conjectures through the use of their intuition or the use of measurement. Of course, many originals are provided in which students can discover proofs for themselves.

As to certain specific proposals of the Commission: Space geometry is included. There are many two- and three-page sections of such material appropriately placed throughout the text. In the first chapters of the book, solid geometry exercises are mixed with plane geometry exercises. It would have been ideal if such a scheme could have been continued throughout the text. In the early chapters much of the solid geometry is developed intuitively, and in the last chapters there are many opportunities for deductive development.

Coordinate geometry is introduced. Most of such content occurs after the sequence of theorems includes the Pythagorean relation. Some attempt is made to correlate coordinate geometry with synthetic geometric topics. However, the correlation is not so successful as the correlation of plane geometry with solid geometry. The coordinate geometry deals with slope of lines, distance between points, and locus of

points. Only a few opportunities are provided for analytic geometry proofs of theorems.

Additional innovations need to be written into the text before one can say the Commission's recommendations have been followed. But, in general, the authors have made quite a noble attempt to design a text that will help teachers give students a more significant course. Innovations have been made, without departing from the over-all sequence with which geometry teachers are familiar. For this reason, the text will probably find rather wide acceptance.—*Oscar Schaaf, University of Oregon, Eugene, Oregon.*

Solid Geometry, A. M. Welchons, W. R. Krickenberger, and Helen R. Pearson (Boston, Ginn and Company, 1959). Cloth, v+346 pp., $3.48.

The authors state in the preface that this text is a revision of their *New Solid Geometry*, published in 1955. They are quite correct in further stating that the "general plan and sequence of topics have remained much the same."

The major changes do not seem to be in the area of subject matter or approach to solid geometry, but rather are in the areas of page format and use of color.

The individual pages of the book are eye-catching because of the arrangement of proofs, the improvement in the drawings, the use of color, and the increased size of the page. These attractive qualities would seem to stimulate individual study on the part of the student using the text.

Color has been used in an intelligent attempt to improve the figures in the text, and the topics on locus have been greatly enhanced by the introduction of color. This should help the student who is weak in spatial perception.

The text is a long one, some 325 pages, not including comprehensive tests and tables at the end of the book. There are sections devoted to Spherical Trigonometry, an Introduction to Coordinate Geometry of Space, and Space Perception. Despite the authors' statements in the preface, this reviewer feels that the book tends to promote a full semester course in solid geometry. If schools prefer this, then here is an excellent text in the traditional sense.

It is felt that the text is not along the lines of current trends in teaching solid geometry, at least not those trends suggested by groups currently studying curriculum revisions. Most of these groups suggest a fusion of the plane and solid geometry, with a high degree of intuition being used in the solid geometry. This text is not written to these suggestions.

Because of the clarity of its figures, the intelligent use of color, and the eye-catching quality of its pages, this text is an improvement over most traditional texts on solid geometry.—*Edwin C. Douglas, The Taft School, Watertown, Connecticut.*

Succeeding in College Entrance Tests, Joseph R. Orgel, Julius Freilich, and Simon L. Berman (New York: Oxford Book Company, 1959). Paper, 314 pp.

Primary emphasis in this book is given to the *Scholastic Aptitude Test* (SAT) which is required of each student who takes a College Entrance Examination. The purpose of the book is modest. The authors hope that the examinee will, after proper use of their book, do his best on the test. He should accomplish this, they feel, because he has been familiarized with the examination experience ahead of him. He is told just what the SAT is, how it is administered, and is even given several sample tests (many with solutions and all with answers). This is supposed to relieve test "jitters" that stem from entering into an unfamiliar experience. They also hope to build his confidence by giving him experience with certain types of problems he may meet. Included in the book is a brief review of mathematics.

The CEEB has recently completed several studies pertaining to "coaching" on the SAT and has concluded "that intensive drill for the SAT does not yield gains in scores large enough to affect decisions made by colleges with respect to the admission of students."[1] However, the board points out that on the mathematics section of this test, efforts along this line seem to raise the scores an average of 25 points on a 600-point scale.[2]

The CEEB studies have pointed out that the scores of students who are left alone change in the same way as the scores of tutored students. In fact "about 1 student in 15 will find that his scores increase by 100 points or more between junior and senior years in high school, *and this is true whether he is coached or not.*"[3]

The CEEB is concerned about the expense many parents of modest means take on to purchase tutoring for the SAT. The Board is also sensitive to parental feeling that an unfair advantage has gone to those able to afford and receiving tutoring. The availability of this type of book should help to alleviate this problem.

Among the specimen test items, the reviewer finds problems containing insufficient information. A student, realizing this, might logically choose the response "none of these" (answers). It may be argued that upon looking ahead at the answers at the end of each test the student could easily translate the question in the manner expected by the authors. The reviewer, however, doubts that such aid in translating questions is available to students actually taking the SAT. Examples of such items are problem 24, page 108, problem 1, page 126, and problem 5, page 121.

On page 121 the authors review per cent. Three times on this one page they stress that

[1] College Entrance Examination Board, New York 27, N. Y., May, 1959—Number 5.
[2] *Ibid.*
[3] *Ibid.*

"per cent" (or %) means "divided by 100." The first model problem is "By what % is 30 more than 24?" Do the authors intend this problem to be translated, "By what divided by 100 is 30 more than 24"? A few lines lower on the same page the sentence "$2\frac{1}{2}\% = 140$" appears. According to the authors' definition of "%" this sentence means: $2\frac{1}{2} \div 100 = 140$. In the reviewer's opinion, calling this "sloppy language" would be much too kind.

In many cases the authors give "short cut" solutions to model problems. These in many cases could be improved upon. For instance model problem 11, page 111, asks: "Which fraction has a value closest to $\frac{1}{2}$? $\frac{4}{7}$, $\frac{4}{9}$, $\frac{4}{5}$, $\frac{4}{3}$." The solution offered is to "convert each fraction into its decimal equivalent."

It seems to this reviewer that getting an equivalent fraction for each given fraction with the denominator 36 would be much easier. The solution the authors suggest for problem 20, page 180, takes seven steps. The problem: "It takes Amy 30 minutes to type a certain report, but May can do it in 20 minutes. How long would it have taken them had they worked together?" The reviewer suggests this solution: Suppose they worked together an hour. They would have completed 5 reports. So it would take them 12 minutes for 1 report.

In the solution to problem 3, page 192, the authors go from "$x^4 = 16$" to "$x = \sqrt[4]{16}$." These equations are not equivalent to this reviewer.

This book does contain a wealth of specimen items. However, it would seem from the items displayed that there is no trend toward introducing problems calling for "modern mathematics" in the SAT.—*Herbert Wills, York Community High School, Elmhurst, Illinois.*

Technical Calculus and Analysis, Harold S. Rice and Raymond M. Knight (New York: McGraw-Hill Book Company, Inc., 1959). Cloth, ix+460 pp., $6.75.

This text was written for students in technical institutes, junior colleges, and industrial training programs. It contains seven chapters and an appendix. The chapter titles are: "Graphical Methods," "Empirical Equations," "Differentiation," "Integration," "Centroids and Moments," "Introduction to Differential Equations," "Probability and Statistics." Calculus is treated primarily as a tool to solve engineering problems, and the book has a great wealth of problems. The chapter on centroids and moments is an unusually long one—103 pages.

The limit concept is discussed briefly in the text and in the appendix; it is not treated mathematically. Continuity, multiple integration, and partial differentiation receive brief mention in the appendix. There is no treatment of infinite series, power series, and expansion of functions. Differentiation and integration techniques are presented through a development which ends in rules stated in words instead of the usual formulas in bold-faced type. Many of the exercises are based on realistic industrial problems and require considerable computation and tabulation. Students are encouraged to use handbooks and tables of integrals. Formulas for differentiating $\sin x$, $\cos x$, and e^x are developed using the power series expansions. A footnote refers to another text for these expansions. Term-by-term integration and differentiation are accomplished without justification or apology.

On page 8, it is implied that an area formed by four lines, $x = a$, $x = b$, $y = mx + b$, and $y = 0$, is either a triangle or a trapezoid. This overlooks such an example as the one determined by $x = 1$, $x = -1$, $y = x$, and $y = 0$. On page 17 "there is then a critical position of the secant DE where the chord is zero." Perhaps "where the chord is zero in length" might be better. Perhaps, "where the chord has shrunk to a point." On page 362 there appears, "Differentiating Eq. (5), we obtain. . . ." The reviewer prefers to differentiate functions, and would suggest, "From Eq. (5) we obtain by differentiation. . . ."

Students who work a reasonable number of the exercises in this text undoubtedly will learn to use calculus as a tool in solving a variety of applied problems. The aim of the authors was to produce a text for classes of students who will work in the technological areas of production and construction rather than in basic engineering research and design. This reviewer believes that they have achieved this aim.—*Lawrence A. Ringenberg, Eastern Illinois University, Charleston, Illinois.*

Understanding and Teaching Arithmetic, E. T. McSwain and Ralph J. Cooke (New York, Henry Holt and Company, 1958). Cloth, xi+420 pp., $5.50.

The publication of this volume is without a doubt the result of a great deal of intensive analysis of the field of arithmetic and an exploration into the underlying meanings of the concepts involved. The book contains many sections which should prove to be very helpful to a teacher who is intent on taking the subject out of the area of rote or mechanical learning. These features include:

1. exploratory questions which focus one's thinking on crucial issues
2. diagnostic questions which might well be used to determine areas where reteaching is necessary
3. practice and self-evaluation exercises for the student preparing for teaching
4. suggested, purposeful activities suitable as projects for teacher education classes
5. concise summary statements at chapter endings
6. well-chosen references at chapter endings for use in collateral reading.

The over-all content covers the usual topics found at all grade levels, with a somewhat less intensive treatment of applications in the fields of business, industry, and measurement. Concluding chapters give particular emphasis to problem solving, evaluation, and curriculum considerations, such as basic philosophy, sequence, and meeting the needs of the gifted children.

The reader of this volume cannot help but be impressed with its paramount objective—to expose the pupil to mathematical meanings. Some of the presentations are common property of most similar volumes. Unique to this treatment, however, is an extreme extension of the implications of place value, particularly in the processes of multiplication and division, and much more especially with the involvement of decimals. Illustrative is the following teaching example in multiplication, with the accompanying explanation (page 83):

H	*T*	*O*
	2	0
	3	0
		0
	0	
6	0	
6	0	0

"In example (a), 30×20, the question asked and to be answered is, 'What number equals 30 (3 tens) groups of 2 tens each or 3 tens×2 tens=?' By thinking of 20 as 2 tens and 0 ones and applying the principle of computing with like units, first ones and then tens, the steps are: Step 1—think, 'The multiplier consists of 3 tens and 0 ones. First multiply by the ones. There are no ones in the multiplier. Write 0 in the ones place in the first partial product.' Step 2—think, 'Now multiply by the tens. There are no ones to multiply by 3 tens. Write 0 in the tens place in the third partial product.' Step 3—think, 'What number equals 30 (3 tens) groups of 2 tens each?' The answer is 60 tens. Write 6 in the hundreds place in the fourth partial product. The total product in example (a) is the sum of 6 hundreds, 0 tens, and 0 ones or 600."

The following illustration is presented, among several others, as a preferred method of teaching the multiplication of decimals:

.246	
×38.	
.048	8 ones×.006
.32	8 ones×.04
1.6	8 ones×.2
.18	3 tens×.006
1.2	3 tens×.04
6.	3 tens×.2
9.348	Product

Analogous procedures requiring thought processes involving decimals at each step are suggested for examples in division of decimals. Space does not permit giving additional illustrations.

It is the writer's feeling that the procedures described above, as well as others presented elsewhere in the volume, are too complex for the average child, and that the insistence that meanings be extended to this point might well give occasion for errors not encountered in methods as ordinarily taught.

Perhaps the greatest contribution that this volume can make is to challenge the teacher to become more alert to the importance of meanings. General acceptance of some of the teaching suggestions might well be deferred, at least until they have been tried out empirically with children of all levels of ability, or until their efficacy has been established through carefully conducted controlled experiments.

There are a few evidences that the original manuscript was not subjected to critical appraisal before publication. For example, the following arithmetic operation appears as a partial explanation of a division example in decimals: $2\times.004=.008+.25\times.004=.008+.001=.009$.

In another place, the presentation of the determination of the formula for the area of a trapezoid requires an understanding of algebraic processes not normally comprehended in the upper elementary grades. The writer feels that other more suitable methods should have been presented.—*Harry C. Johnson, University of Minnesota, Duluth Branch, Duluth, Minnesota*

"Mathematics is the predominant science of our time; its conquests grow daily, though without noise; he who does not employ it *for* himself, will one day find it employed *against* himself."—*J. F. Herbart*, Werke, *Langensalga, 1890, Bd. 5, p. 105.*

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Annual financial report

Below is a brief financial report of the Council for the fiscal year ending May 31, 1959. The reader will note that from a financial viewpoint the year was a good one for the Council.

No doubt the most striking feature of the business of the past year was the continued rapid increase in membership in spite of an increase in dues. Nearly all professional organizations have experienced a temporary decline in membership or in membership growth after an increase in membership dues. By contrast the membership of the Council increased at the greatest rate in its history.

Another noteworthy development was an increase of about 50 per cent in the sale of publications over the sales of the previous year. These two factors produced unexpected income which has greatly strengthened the Council financially.

Receipts and expenditures of The National Council of Teachers of Mathematics for the fiscal year, June 1, 1958-May 31, 1959

June 1, 1958—Total cash resources		$53,345.08
RECEIPTS		
Memberships with THE MATHEMATICS TEACHER subscriptions	$ 72,258.73	
Memberships with *The Arithmetic Teacher* subscriptions	23,351.48	
Institutional subscriptions to THE MATHEMATICS TEACHER	27,147.82	
Institutional subscriptions to *The Arithmetic Teacher*	20,715.29	
Subscriptions to *The Mathematics Student Journal*	18,174.94	
Sale of advertising space in THE MATHEMATICS TEACHER	7,260.08	
Sale of advertising space in *The Arithmetic Teacher*	1,473.07	
Rental of membership list	2,520.31	
Interest on U. S. Treasury Bonds	375.00	
Net profit from conventions	3,292.58	
Miscellaneous	461.66	
Sale of publications		
Yearbooks	35,828.92	
Supplementary publications, etc.	24,198.57	
Total receipts	$237,058.45	
EXPENDITURES		
Washington office	$ 65,219.31	
Purchase of Addressograph equipment	5,203.43	
President's office	2,849.35	
Vice-Presidents' office expenses	125.00	
THE MATHEMATICS TEACHER	38,083.65	
The Arithmetic Teacher	14,083.50	
The Mathematics Student Journal	12,179.75	
Secondary School Curriculum Committee	5,432.02	
Elementary School Curriculum Committee	898.58	
Committee work	3,194.14	
Travel by Board members	4,413.56	
Preparation and printing of yearbooks	20,691.53	
Preparation and printing of supplementary publications	8,202.53	
Distributing special publications	3,724.54	
Printing and mailing *Membership Directory*	5,669.30	
Storage and shipment of publications, miscellaneous	853.98	
Total expenditures	$190,824.17	
Increase in cash resources		$46,234.28
May 31, 1959—Total cash resources		**$99,579.36**

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Operating Committees (1959-60)

SECRETARY OF THE BOARD

Houston T. Karnes, Baton Rouge, La., 1960

NATIONAL COUNCIL REPRESENTATIVES

AAAS Cooperative Committee on Science and Mathematics
Bruce Meserve, Montclair, N. J., 1960

Educational Advisory Committee to Science Service
Veryl Schult, Washington, D. C., 1960
Helen Cooper, Bethesda, Md., 1961

Conference Board of the Mathematical Sciences
Burton Jones, Boulder, Colo., 1961
John Mayor, Washington, D. C., 1960

United States Commission on Mathematics Instruction
Howard F. Fehr, New York, N. Y., 1960
Phillip Jones, Ann Arbor, Mich., 1961

AAAS Council
Phillip Jones, Ann Arbor, Mich., 1961
Veryl Schult, Washington, D. C., 1961

EXECUTIVE, 1960

Harold Fawcett, Columbus, O.
Phillip Jones, Ann Arbor, Mich.
Philip Peak, Bloomington, Ind.

PROBLEM-POLICY, 1960

Howard F. Fehr, New York, N. Y., Chairman
John R. Mayor, Washington, D. C.
Phillip Jones, Ann Arbor, Mich.
Philip Peak, Bloomington, Ind.

BUDGET

Houston T. Karnes, Baton Rouge, La., Chairman, 1960
Bruce Meserve, Montclair, N. J., 1962
Vernon Price, Iowa City, Iowa, 1961

Duties: To prepare the operating budget of the Council for the year 1960–61 and to present it to the Board at its annual meeting in Buffalo.

NOMINATIONS AND ELECTIONS, 1960

Ida Bernhard Puett, Atlanta, Ga., Chairman
Milton Beckmann, Lincoln, Neb.
Charles Butler, Kalamazoo, Mich.
Howard F. Fehr, New York, N. Y.
Kenneth Henderson, Urbana, Ill.
Houston T. Karnes, Baton Rouge, La.
Albert Linton, Philadelphia, Pa.
Ann Peters, Keene, N. H.
Oscar Schaaf, Eugene, Ore.

Duties: To provide for the spring election in 1960 a slate of nominees all of whom have indicated their willingness to serve if elected.

NOMINATIONS AND ELECTIONS, 1961

Oscar Schaaf, Eugene, Ore., Chairman
Harold Fawcett, Columbus, O.
W. Eugene Ferguson, Newton, Mass.
Lenore John, Chicago, Ill.
Houston T. Karnes, Baton Rouge, La.
W. C. Lowry, Charlottesville, Va.
Ida Bernhard Puett, Atlanta, Ga.
Max Sobel, Fair Lawn, N. J.
Lottchen Hunter, Wichita, Kan.

Duties: To provide for the spring election in 1961 a slate of nominees all of whom have indicated their willingness to serve if elected.

NOMINATION OF EDITOR FOR THE ARITHMETIC TEACHER, 1960

Maurice Hartung, Chicago, Ill., Chairman
Irene Sauble, Detroit, Mich.
Ben Sueltz, Cortland, N. Y.

Duties: To nominate two persons, each of whom has indicated his willingness to serve as editor of *The Arithmetic Teacher* if elected by the Board, and to present these nominees to the Board at its 1959 summer meeting at Ann Arbor.

REPORTING ELECTIONS, 1960

Ida Bernhard Puett, Atlanta, Ga., Chairman
Myrl Ahrendt, Washington, D. C.
Harold Fawcett, Columbus, O.

Duties: To certify the results of the annual election to membership on the Board of Directors and to report these results at the annual business meeting.

Affiliated Groups

Eugene Smith, Wilmington, Del., Chairman, 1961
Northeastern: Catherine Lyons, Pittsburgh, Pa., 1960
North Central: Virginia Pratt, Omaha, Neb., 1961
Western: Kenneth Skeen, Concord, Cal., 1961
Southeastern: Houston Banks, Nashville, Tenn., 1960
Central: Adeline A. Riefling, St. Louis, Mo., 1961
Southwestern: Eunice Lewis, Norman, Okla., 1961

Membership, 1960

Mary Rogers, Westfield, N. J., Chairman
Myrl Ahrendt, Washington, D. C.
Marian C. Cliffe, Glendale, Cal.
Mary Rickey, Cedar Rapids, Iowa
Eugene Smith, Wilmington, Del.

Duties: To suggest ways and means of increasing membership in the Council and to translate these suggestions into action through co-operative work with the Affiliated Groups and the State Representatives.

Place of Meeting

Alice Hach, Racine, Wis., Chairman, 1960
Mabel Baker, Pittsburgh, Pa., 1961
Forest N. Fisch, Greeley, Colo., 1960
Eugene Nichols, Tallahassee, Fla., 1961
James Nudelman, Cupertino, Cal., 1961
Myrl Ahrendt, Washington, D. C., ex officio

Duties: To study, plan, and report to the Board concerning geographic and strategic locations of all Council meetings and to recommend to the Board a planned sequence of convention cities through 1966.

Research

John Kinsella, New York, N. Y., Chairman, 1962
Kenneth Brown, Washington, D. C., 1961
Howard F. Fehr, New York, N. Y., 1961
Sheldon Myers, Princeton, N. J., 1962
Fred Weaver, Boston, Mass., 1961

Duties: To promote research in mathematics education, to provide a means of summarizing and publishing significant research in this area, and to plan the program for a Research Section at the annual meeting.

The Mathematics Teacher, 1960

Robert Pingry, Urbana, Ill., Editor, 1962
Jackson Adkins, Exeter, N. H.
Mildred Keiffer, Cincinnati, O.
Daniel Lloyd, Washington, D. C.
Z. L. Loflin, Lafayette, La.
Ernest Ranucci, Newark, N. J.

The Arithmetic Teacher, 1960

Ben A. Suelts, Cortland, N. Y., Editor, 1960
Marguerite Brydegaard, San Diego, Cal.
John R. Clark, New Hope, Pa.
Glenadine Gibb, Cedar Falls, Iowa
Joseph J. Urbancek, Evanston, Ill.

The Mathematics Student Journal, 1960

W. W. Sawyer, Middletown, Conn., Editor, 1961
Arnold Ross, Notre Dame, Ind.
Oscar Schaaf, Eugene, Ore.

Supplementary Publications

L. A. Ringenberg, Charleston, Ill., Chairman, 1960
Houston Banks, Nashville, Tenn., 1960
Marguerite Brydegaard, San Diego, Cal., 1961
Edwin Eagle, San Diego, Cal., 1960
Kenneth Henderson, Urbana, Ill., 1962
Burton Jones, Boulder, Colo., 1961
Margaret Joseph, Milwaukee, Wis., 1962
Jesse Osborn, St. Louis, Mo., 1961
Mildred Keiffer, Cincinnati, O., 1960
Helen A. Schneider, Oak Park, Ill., 1961
H. C. Trimble, Cedar Falls, Iowa, 1961
James Ulrich, Arlington Heights, Ill., 1960

Duties: To plan a program of small publications in addition to the three journals and the yearbooks, to solicit and evaluate manuscripts received, and to submit to the Publications Board those which seem acceptable for publication.

Publications Board

Clifford Bell, Los Angeles, Cal., Chairman, 1960
Henry Swain, Winnetka, Ill., Chairman, 1961
Henry Van Engen, Madison, Wis., Chairman, 1962
The Mathematics Teacher, Robert Pingry, 1962
The Arithmetic Teacher, Ben Sueltz, 1960
The Mathematics Student Journal, W. W. Sawyer, 1961
Supplementary Publications, L. A. Ringenberg, 1960
Yearbook Planning Committee, Bruce Meserve, 1961

Duties: To propose and coordinate publication policies of the Council, to make recommendations to the Board concerning major publications, and to report action on minor publications to the President.

Yearbook Planning

Bruce Meserve, Montclair, N. J., Chairman, 1961
Houston Banks, Nashville, Tenn., 1962
Robert Pingry, Urbana, Ill., 1960

Duties: To report on status of yearbooks in progress, to recommend topics for future yearbooks, and to suggest possible editors and committee members.

YEARBOOKS

25th Arithmetic

Foster Grossnickle, Jersey City, N. J., Chairman
Dan Dawson, Stanford, Cal.
Ida Mae Heard, Lafayette, La.
Irene Sauble, Detroit, Mich.
Herbert Spitzer, Iowa City, Iowa
Louis C. Thiele, Detroit, Mich.

26th Evaluation

Donovan Johnson, Minneapolis, Minn., Chairman
Robert Fouch, Tallahassee, Fla.
Glenadine Gibb, Cedar Falls, Iowa
Max Sobel, Fair Lawn, N. J.
Ben A. Sueltz, Cortland, N. Y.
Fred Weaver, Boston, Mass.

ELEMENTARY SCHOOL CURRICULUM

Fred Weaver, Boston, Mass., Chairman, 1961
Joyce Benbrook, Houston, Tex., 1960
Laura Eads, New York, N. Y., 1960
Ann Peters, Keene, N. H., 1962
Irene Sauble, Detroit, Mich., 1961
Henry Van Engen, Madison, Wis., 1962

SECONDARY SCHOOL CURRICULUM, 1960

Frank B. Allen, La Grange, Ill., Chairman
Jackson B. Adkins, Exeter, N. H.
Harold Fawcett, Columbus, O.
Howard F. Fehr, New York, N. Y.
A. S. Householder, Oak Ridge, Tenn.
Lottchen Hunter, Wichita, Kan.
Burton Jones, Boulder, Colo.
John R. Mayor, Washington, D. C.
Bruce Meserve, Montclair, N. J.
Sheldon Myers, Princeton, N. J.
E. B. Newell, Indianapolis, Ind.
Alfred Putnam, Chicago, Ill.
Elizabeth Roudebush, Seattle, Wash.
Helen Walker, New York, N. Y.
Marie Wilcox, Indianapolis, Ind.
Lynwood Wren, San Fernando, Cal.
Magnus Hestenes, Los Angeles, Cal. (Advisor)

TEACHER EDUCATION AND CERTIFICATION

Henry Syer, Kent, Conn., Chairman, 1961
Charles Atherton, Shepherdstown, W. Va., 1962
Kenneth Brown, Washington, D. C., 1960
Daniel Lloyd, Washington, D. C., 1961
Robert Kalin, Tallahassee, Fla., 1962
David Page, Urbana, Ill., 1960
Richard Purdy, San Jose, Cal., 1962

Duties: To study the present certification requirements for teachers of mathematics in the respective states including the mathematical background of elementary school teachers and to recommend appropriate policy to the Board of Directors.

TELEVISION

Joseph Hooten, Tallahassee, Fla., Chairman, 1960
George Anderson, Millersville, Pa., 1960
Emil Berger, St. Paul, Minn., 1961
Mary Nesbitt, Miami, Fla., 1962
David Wells, Pontiac, Mich., 1960

Duties: To study the status of television in the teaching of mathematics and to propose policy to the Board of Directors concerning the use of this medium in mathematics education.

MATHEMATICS FOR THE TALENTED, 1960

Julius H. Hlavaty, New Rochelle, N. Y., Chairman
Mary Lee Foster, Arkadelphia, Ark.
Frances Johnson, Oneonta, N. Y.
Glen Vanatta, Indianapolis, Ind.
Robert Fouch, Tallahassee, Fla.
Joseph Payne, Ann Arbor, Mich.
Harry Ruderman, New York, N. Y.

Duties: To develop plans and proposals designed to improve and strengthen the mathematics education of talented students.

COOPERATION WITH INDUSTRY, 1960

Marie Wilcox, Indianapolis, Ind., Chairman
Jesse Cardwell, Dallas, Tex.
William Glenn, Pasadena, Cal.
Kenneth Kidd, Gainesville, Fla.
Zeke Loflin, Lafayette, La.
G. A Rietz, New York, N. Y.
Lauren Woodby, Mt. Pleasant, Mich.

Duties: To publish in the official journals of the Council descriptions of cooperation between industry and mathematics education and to plan the program for a "Cooperation with Industry" section at the annual meeting.

INTERNATIONAL RELATIONS

E. H. C. Hildebrandt, Evanston, Ill., Chairman, 1962
Ernest Ranucci, Newark, N. J., 1961
Alfred Putnam, Chicago, Ill., 1961
Veryl Schult, Washington, D. C., 1960

Duties: To extend the hospitality and services of the Council to mathematics teachers and students from the United States who are visiting other countries and to foreign mathematics teachers and students in this country, to cooperate with international organizations in securing an interchange of information concerning mathematics education, and to plan the program for an "International Section" at the annual meeting.

Report of the Policy Conference of the National Council of Teachers of Mathematics

Conrad Hilton Hotel, Chicago

October 9–10, 1959

About thirty leaders in mathematics teaching met in Chicago, October 9–10, 1959. The main purpose of this meeting was to consider the role of The National Council of Teachers of Mathematics in the changing picture of mathematics education at the present and for the future. A report of this conference will be published in the near future.

President Fawcett opened the conference by making the following statement:

The National Council of Teachers of Mathematics was established in February, 1920, its stated purpose being "to assist in promoting the interest of mathematics in America, especially in the elementary and secondary fields." Since its creation nearly 40 years ago it has been guided by this purpose and during this period its services to both elementary and secondary teachers of mathematics have steadily increased. Within the last decade or so, developments on a world front have provided a long overdue awakening to the great importance of mathematics in the educational program of our young people and today the improvement of mathematics education from kindergarten through college is a problem which has commanded the active interest of many professional organizations and is of concern to all thoughtful people. Significant curricular studies are in process. New instructional materials are being developed. An in-service program of large proportions is now operating and the collegiate program for the mathematics teachers of tomorrow is under revision. Large financial assistance is available for all of these projects and it is evident that the original purpose of the Council is now receiving wide support. What then should be the role of the Council in this changed intellectual climate? The School Mathematics Study Group, for example, is developing curricular materials for grades 7 through 12. On the Illinois campus, studies are in process which involve the mathematics curriculum on both the elementary and secondary levels. The Council has an Elementary School Curriculum Committee and a Secondary School Curriculum Committee. What should be the relation of these committees to the work of these groups? How can we most effectively co-operate with all such groups in promoting the interests of mathematics in America?

Mathematics teachers throughout the country are awakening to the important fact that mathematics is a live and growing subject. They have heard of the *new mathematics* and they turn to the National Council for guidance and help in connection with their own local programs. The increasing volume of correspondence received by our Washington office, the steady increase in attendance at regular meetings of the Council, and the rapid growth in our membership reflect the confidence of mathematics teachers in the organization. During the past five years, for example, the membership of the Council has approximately doubled. The rate of increase has accelerated and during last year 4297 new members were added to the membership rolls. They look to the Council for leadership and one major purpose of this meeting is to suggest services through which the Council can most effectively meet this leadership obligation. Four such services have been proposed by members of the Planning Committee. These will be presented to the conference as indicated in the agenda. A liberal amount of time is then provided for an evaluation of these proposals by members of the conference, for the suggestion of other activities appropriate to the leadership role of the Council, and for ideas as to how those services on which there seems to be substantial agreement can be translated into action. We invite wide participation in the discussion and it is our hope that by discussing these problems together and exchanging our ideas about the future of mathematics education we can collectively arrive at some guide-lines that will help us determine policies which will provide direction for future Council activities to the end that it will serve as a more effective instrument in meeting the needs of its members and all those interested in mathematics education.

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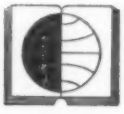

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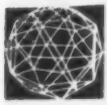

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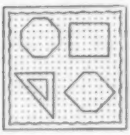

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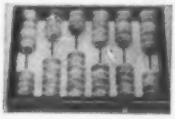

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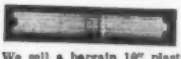

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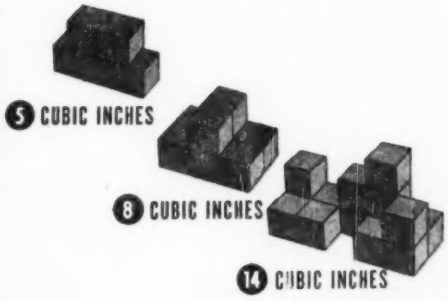

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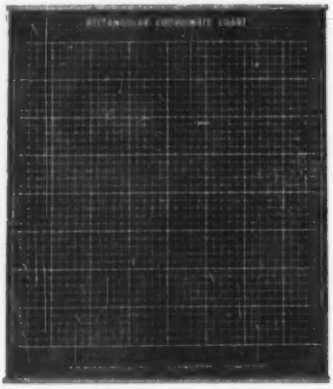

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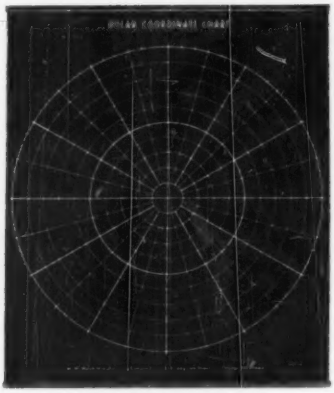
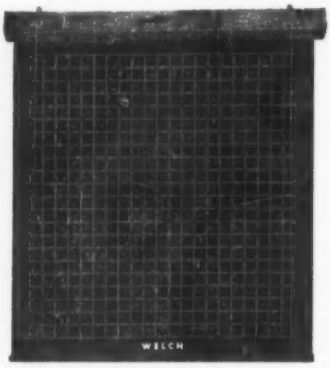


Polar-Coordinate Side.

Use with Regular White or Colored Chalk



No. 0534. Rectangular Coordinates

SLATED-CLOTH TYPE

No. 7068. CHALKBOARD GRAPH CHART, Polar and Rectangular Coordinates. Rectangular Coordinates are on one side and polar coordinates are on the other side of this slated cloth graph chart, accurately ruled. The rectangular graph is 40 x 40 inches, with each inch marked, and the largest circle on the polar chart is 40 inches in diameter, with each 5° and 2 inch radial divisions indicated. Main lines are heavy. On formed-metal chart molding top and bottom ....Each $15.50

STENCIL TYPE

No. 0534. CHALKBOARD GRAPH CHART, Stencil-type, Rectangular Coordinates. This stencil Graph Chart can be hung on the chalkboard chart rail and rubbed with a used eraser to deposit the chalk particles through the perforations. The outline obtained is a clear and distinct rectangular coordinate system. Size 36 x 36 inches of flexible varnished cloth material, on a spring roller and mounted in an oak frame ....Each $23.25

No. 0535. CHALKBOARD GRAPH CHART, Stencil-type, Polar Coordinates. Chart stencil similar to above but Polar Coordinate Design .....Each $23.25

W. M. Welch Scientific Company

DIVISION OF W. M. WELCH MANUFACTURING COMPANY

Established 1880

1515 Sedgwick St. Dept. X Chicago 10, Ill. U.S.A.